Homework 8 – Due Thursday, April 11 at 11:59 PM

Reminder   Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators and write "Collaborators: none" if you worked by yourself. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden. Collaboration is not allowed on problems marked "INDIVIDUAL."

Note   You may use various generalizations of the Turing machine model we have seen in class, such as TMs with two-way infinite tapes, stay-put, or multiple tapes. If you choose to use such a generalization, state clearly and precisely what model you are using. **You may describe Turing machines at a high-level on this assignment.**

Problems   There are 5 required problems and 1 bonus problem.

1. (**Disjoint TMs**) Consider the following computational problem: Given TMs M_1 and M_2, is it the case that $L(M_1) \cap L(M_2) = \emptyset$? This problem is captured by the language $DISJ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) \cap L(M_2) = \emptyset\}$.

   (a) If f is a mapping reduction from A_{TM} to $\overline{DISJ_{\text{TM}}}$, what should the inputs and outputs to f be? (As in Lecture 17, slide 18.)

   (b) Describe a TM computing a mapping reduction from A_{TM} to $\overline{DISJ_{\text{TM}}}$ and explain why this TM is correct.

   (c) Explain why part (b) implies that $DISJ_{\text{TM}}$ is not Turing-recognizable.

   (d) Is there is a mapping reduction from A_{TM} to $DISJ_{\text{TM}}$? Prove your answer.

2. (**Mapping Reductions**) Prove the following general statements about the existence of mapping reductions.

   (a) Let A be an arbitrary recognizable language, and suppose $A \leq_{\text{m}} \overline{A}$. Show that A is decidable. (Hint: Recall that a language is decidable iff both it and and its complement are recognizable.)

   (b) Let A be an arbitrary decidable language and let B be an arbitrary language other than $\emptyset$ or Σ^*. (That is, there exists a string $w_0 \notin B$ and a string $w_1 \in B$.) Show that $A \leq_{\text{m}} B$.

3. (**INDIVIDUAL: Loopy TM**) Consider the following computational problem: Given the description of a Turing machine M, is it the case that M loops forever on *every* input w?

   (a) Formulate this problem as language $LOOP_{\text{TM}}$.

   (b) Prove that $LOOP_{\text{TM}}$ is undecidable. You may do this by either providing an "informal reduction" as on Homework 7, or by giving a mapping reduction – it's your choice. In either case, make sure to explain why the TM computing your reduction is correct.

4. (**Review of asymptotic notation**) Use the formal definitions of O and o notation to prove the following statements. You may also use the fact that $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, as well as assume basic math facts like "$n \leq n^2$ for all $n \geq 1$" without proof.

   (a) Prove that $3n + 2n \log n = O(n \log^2 n)$. (The notation $\log^2 n$ means $(\log n)^2$.)

   (b) Prove that $3^{n^{1/3}} = 2^{o(n)}$.

5. (**Asymptotic notation examples**) This problem will be graded automatically by Gradescope. Please enter your answers manually by completing the assignment Homework 8-Problem 5. For each of the following, select *true* or *false* using the radio buttons on Gradescope. All logarithms are base 2 unless otherwise stated.

   (a) $3^n = 2^{O(n)}$

   (b) $2^{2^n} = O(2^{2n})$

   (c) $n! = O(n^n)$

   (d) $2^n = 3^{o(n)}$

   (e) $2n = o(n^2)$

   (f) $2^{332} = O(n)$

   (g) $16n = O(n)$

   (h) $n^4 = O(n^2 \log n)$

   (i) $n \log^2 n = o(n^2 \log n)$

   (j) $3^n = O(2^n)$

   (k) $2^n = o(3^n)$

   (l) $1 = o(n)$

   (m) $\log(\log n) = o(\log n)$

   (n) $\frac{1}{3} = o(1)$

6. (**Bonus problem**) Show that the assumption that A is recognizable in Problem 2 is necessary: Give an example of an unrecognizable language A such that $A \leq_m \overline{A}$.