BU CS 332 – Theory of Computation

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Lecture 3:

- Deterministic Finite Automata
- Non-deterministic FAs

Reading:

Sipser Ch 1.1-1.2

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Last Time

- Parts of a theory of computation: Model for machines, model for problems, theorems relating machines and problems

- Strings: Finite concatenations of symbols
- Languages: Sets L of strings
- Computational (decision) problem: Given a string x, is it in the language L?

Deterministic Finite Automata

A (Real-Life?) Example

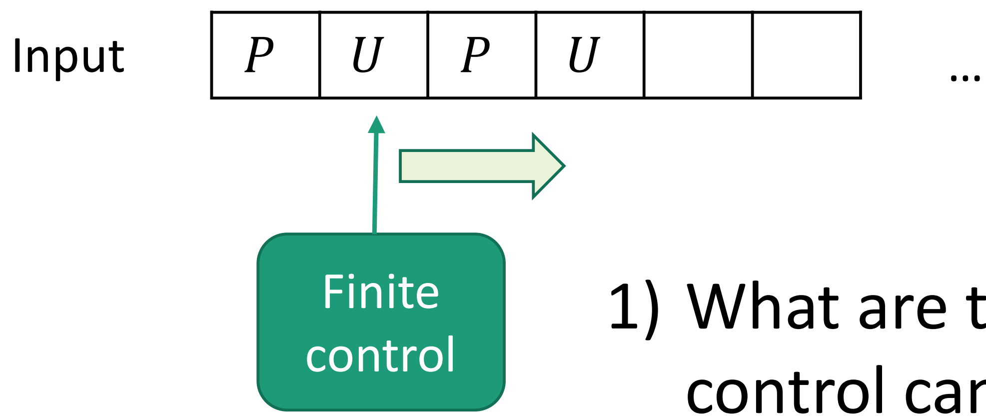
- Example: Kitchen scale
- P = Power button (ON / OFF)
- U = Units button (cycles through g / oz / lb)

  Only works when scale is ON, but units remembered when scale is OFF
- Starts OFF in g mode

- A computational problem: Does a sequence of button presses in $\{P, U\}^*$ leave the scale ON in oz mode?

Machine Models

- <u>Finite Automata (FAs):</u> Machine with a finite amount of unstructured memory



Control scans left-to-right

1) What are the different "states" that the control can be in?
2) In what state does the control start?
3) When the control reads an new input character, how does it transition to a new state?
4) How do I know if I'm in the desired state at the end?

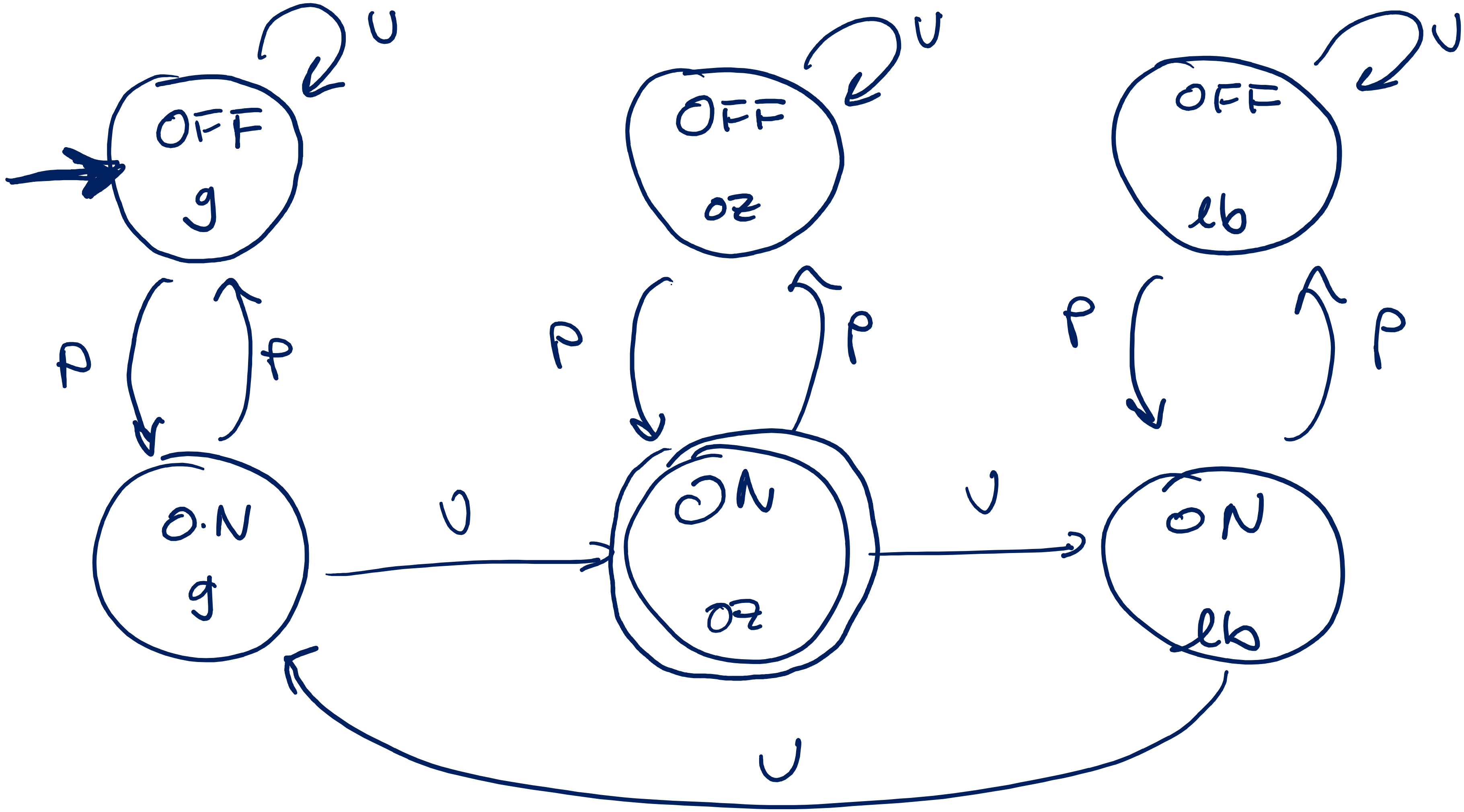
A DFA for the Kitchen Scale Problem

P = Power button (ON / OFF)

U = Units button (cycles through g / oz / lb)

Starts OFF in g mode

<u>Problem:</u> Does a sequence of button presses leave the scale ON in oz mode?



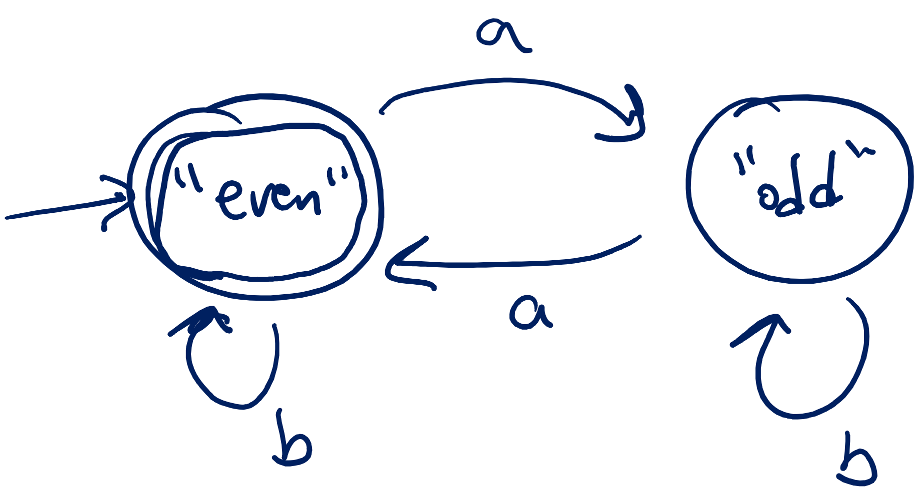
A DFA Recognizing Parity

The language recognized by a DFA is the set of inputs on which it ends in an "accept" state

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{a, b\}$ $L = \{w \mid w$ contains an even number of a's$\}$

even $\xrightarrow{a}$ odd $\xrightarrow{a}$ even $\xrightarrow{b}$ even $\xrightarrow{a}$ odd $\xrightarrow{b}$ odd

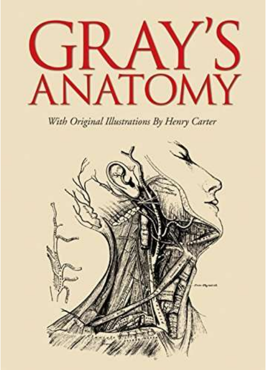


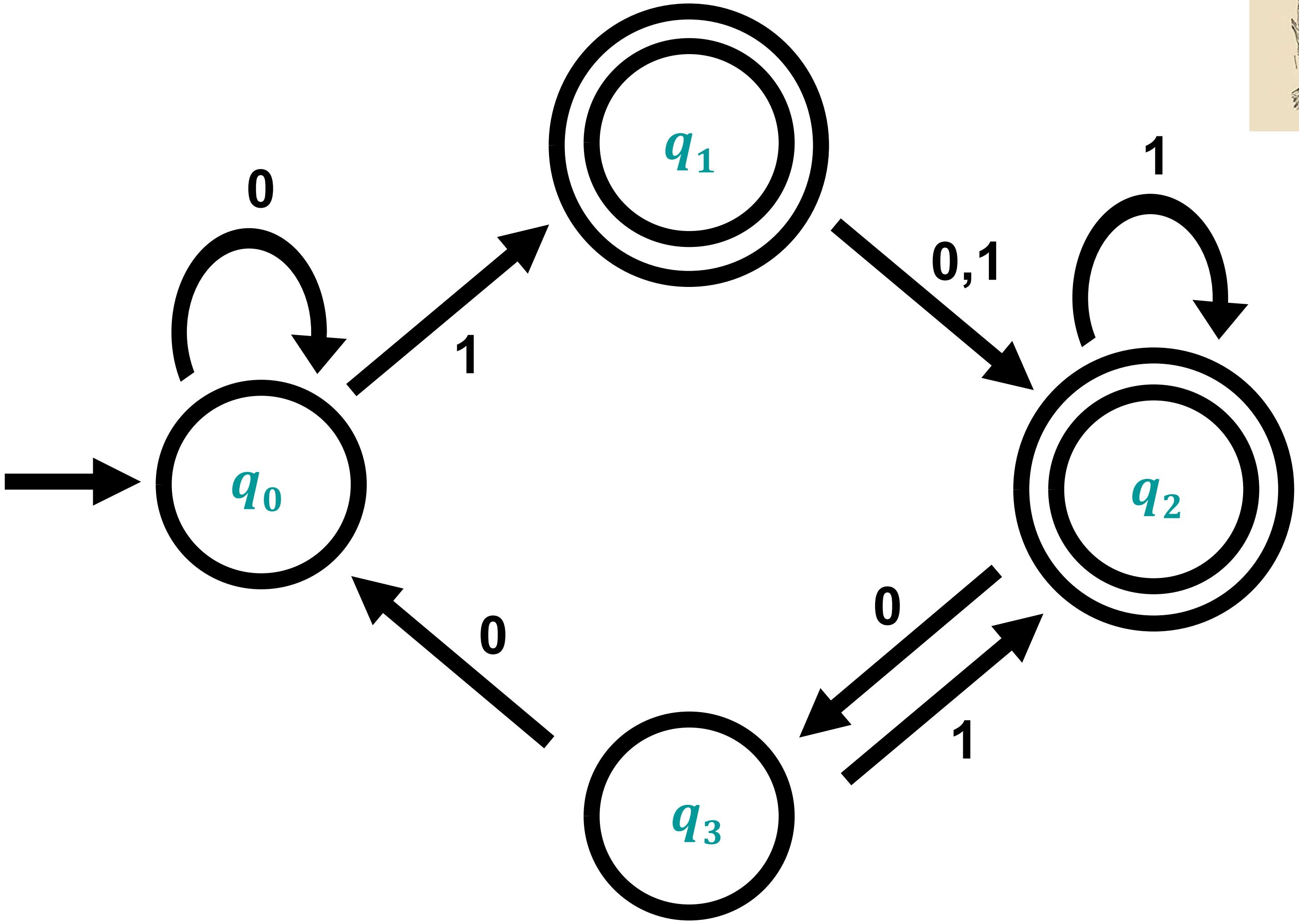
Which state is reached by the parity DFA on input aabab?

a) "even"

b) "odd"

Anatomy of a DFA





Some Tips for Thinking about DFAs

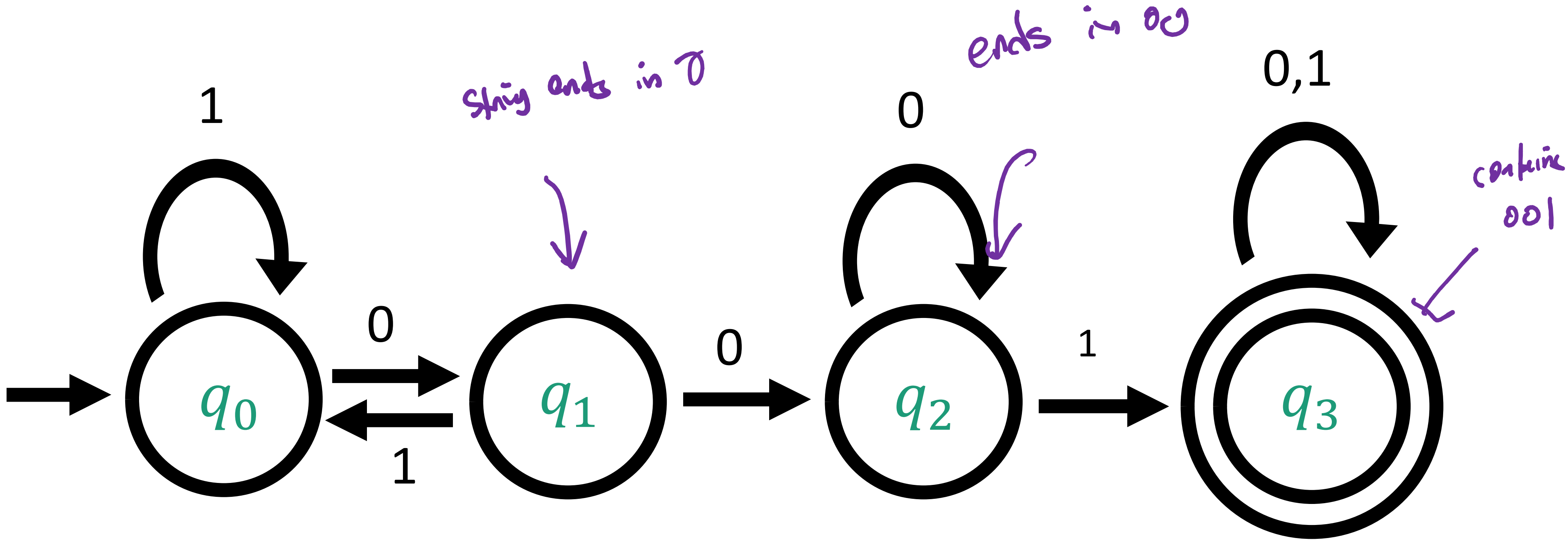
Given a DFA, what language does it recognize?

- Try experimenting with it on short strings. Do you notice any patterns?
- What kinds of inputs cause the DFA to get trapped in a state?

Given a language, construct a DFA recognizing it

- Imagine you are a machine, reading one symbol at a time, always prepared with an answer
- What is the essential information that you need to remember? Determines set of states.

What language does this DFA recognize?



$\{w \in \{0,1\}^* \mid$ w contains 001 as a substring$\}$

$= \{w \in \{0,1\}^* \mid$ w contains two or more 0's followed by a 1$\}$

Practice!

- Lots of worked out examples in Sipser
- Automata Tutor: https://automata-tutor.model.in.tum.de/

Formal Definition of a DFA

A finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states

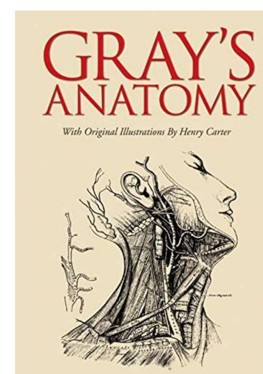
Σ is the alphabet

current state — next symbol — next state

$\delta: Q \times \Sigma \rightarrow Q$ is the transition function

$q_0 \in Q$ is the start state

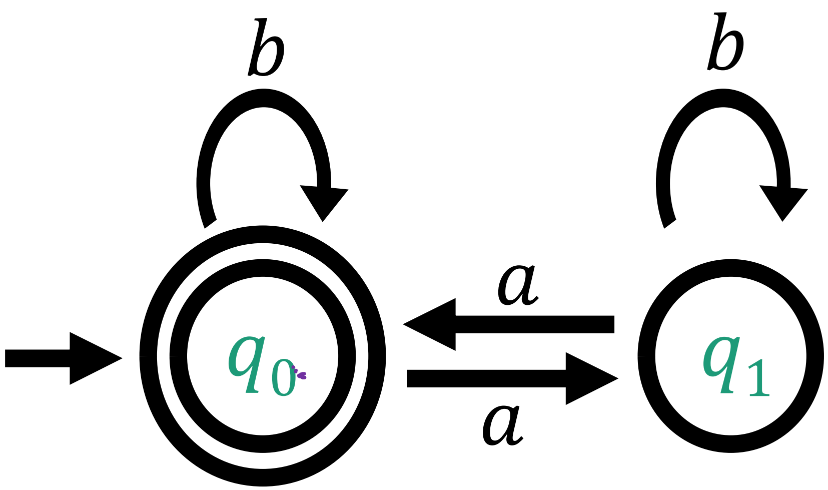
$F \subseteq Q$ is the set of accept states



A DFA for Parity

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{a, b\}$ $\quad$ $L = \{w \mid w$ contains an even number of a's$\}$



State set Q = $\{q_0, q_1\}$

Alphabet Σ = $\{a, b\}$

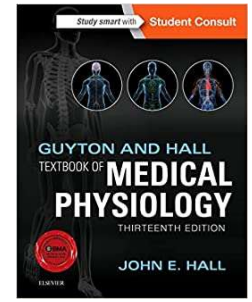
Transition function δ : $Q \times \Sigma \to Q$

δ	a	b
q_0	$\delta(q_0, a) = q_1$	q_0
q_1	q_0	q_1

Start state q_0

Set of accept states F = $\{q_0\}$

Formal Definition of DFA Computation



A DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$

2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for each $i = 0, \ldots, n-1$, and

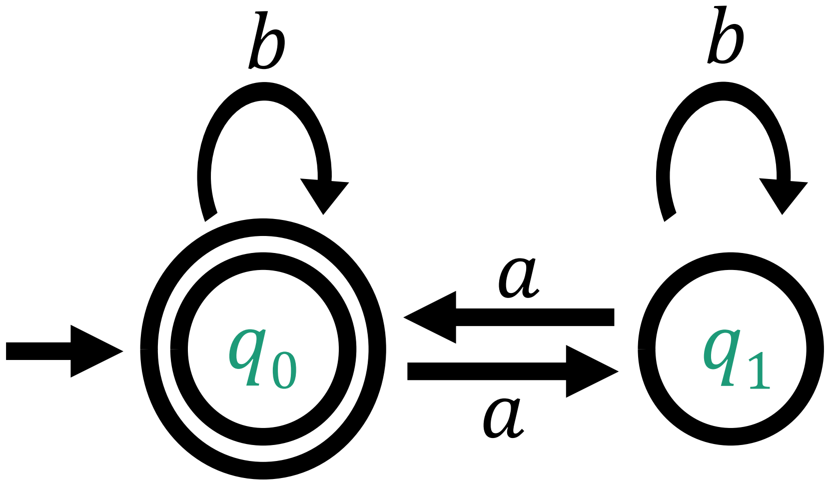
3. $r_n \in F$

$L(M)$ = the language of machine M

= set of all strings machine M accepts

M recognizes the language $L(M)$

Example: Computing with the Parity DFA



$w_1 w_2 w_3 w_4$

Let $w = abba$

Does M accept w?

q_1 b

What is $\delta(r_2, w_3)$?

a) q_0

b) q_1

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ **accepts** a string $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for each $i = 0, \ldots, n-1$
3. $r_n \in F$

$r_0 = q_0$

$r_1 = q_1 = \delta(r_0, w_1) = \delta(q_0, a)$

$r_2 = q_1 = \delta(r_1, w_2) = \delta(q_1, b)$

$r_3 = q_1 = \delta(r_2, w_3) = \delta(q_1, b)$

$r_4 = q_0 = \delta(r_3, w_4) = \delta(q_1, a)$

$r_4 \in F$

Regular Languages

Definition: A language is regular if it is recognized by a DFA

$L = \{ w \in \{a, b\}^* \mid w$ has an even number of a's $\}$ is regular

$L = \{ w \in \{0,1\}^* \mid w$ contains 001 $\}$ is regular

Many interesting problems are captured by regular languages

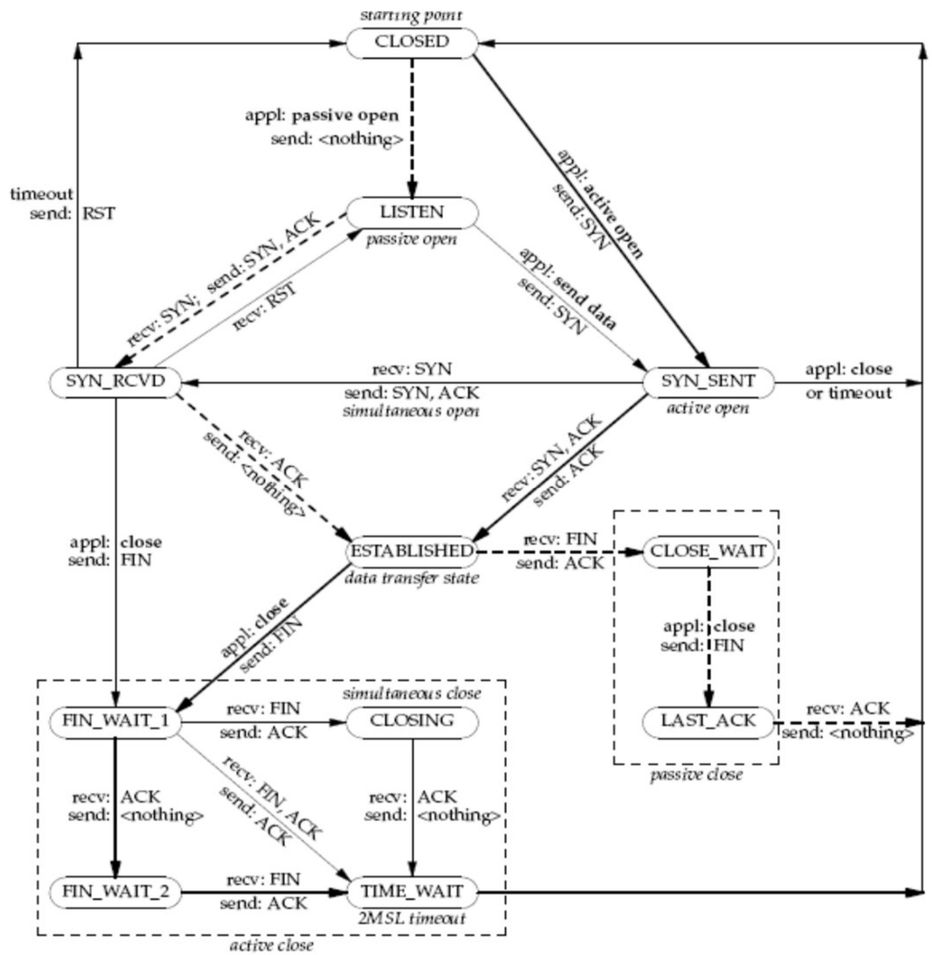
Network Protocols

Compilers

Genetic Testing

Arithmetic

Internet Transmission Control Protocol



Let TCPS = { w | w is a complete TCP Session}

Theorem: TCPS is regular

Compilers

Comments :

Are delimited by /* */

Cannot have nested /* */

Must be closed by */

*/ is illegal outside a comment

$\text{COMMENTS} = \{\text{strings over } \{0,1, /, *\} \text{ with legal comments}\}$

Theorem: COMMENTS is regular

Genetic Testing

DNA sequences are strings over the alphabet $\{\mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{T}\}$.

TAGACAT

A gene g is a special substring over this alphabet.

$g =$ CAT

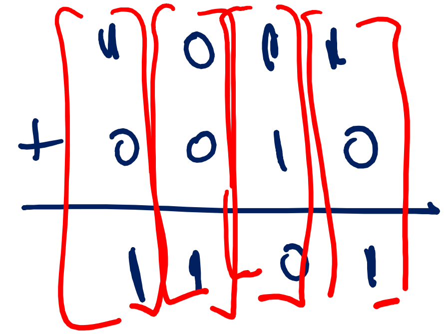
A genetic test searches a DNA sequence for a gene.

GENETICTEST_g = {strings over $\{\mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{T}\}$ containing g as a substring}

Theorem: GENETICTEST_g is regular for every gene g.

Arithmetic

LET $\Sigma = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$



- A string over Σ has three ROWS (ROW_1, ROW_2, ROW_3)
- Each ROW $b_0 b_1 b_2 \ldots b_N$ represents the integer

$$b_0 + 2b_1 + \ldots + 2^N b_N.$$

- Let ADD = $\{S \in \Sigma^* \mid \text{ROW}_1 + \text{ROW}_2 = \text{ROW}_3\}$

Theorem. ADD is regular.

Nondeterministic Finite Automata

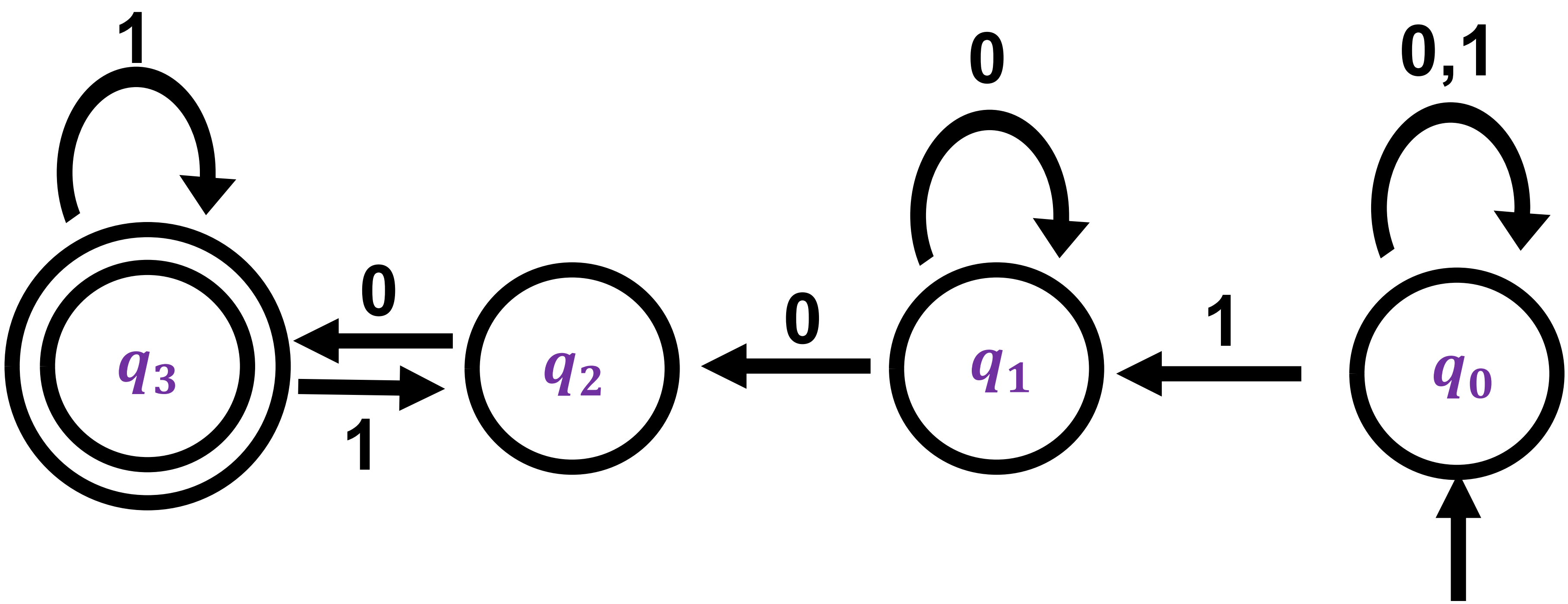
Nondeterminism

In a DFA, the machine is always in exactly one state upon reading each input symbol

In a nondeterministic FA, the machine can "try out" many different ways of reading the same string

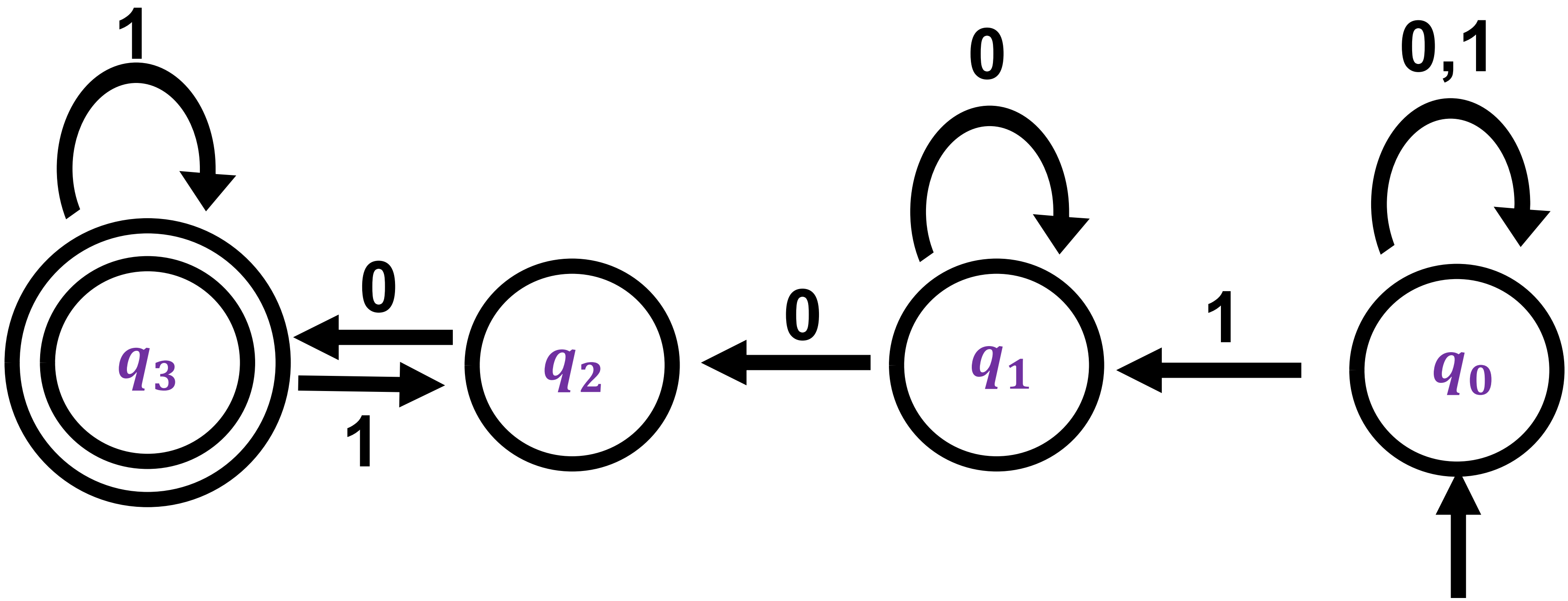
- Next symbol may cause an NFA to "branch" into multiple possible computations
- Next symbol may cause NFA's computation to fail to enter any state at all

Nondeterminism



A Nondeterministic Finite Automaton (NFA) accepts if there ***exists*** a way to make it reach an accept state.

Nondeterminism



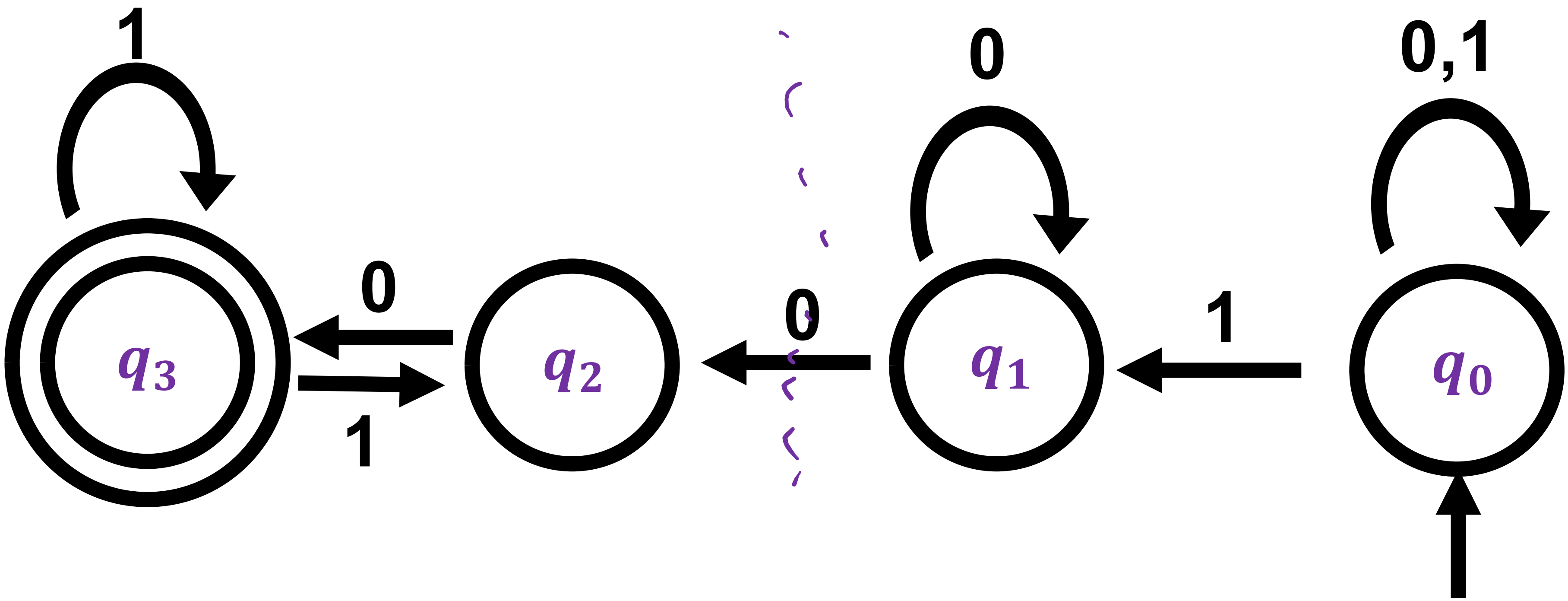
Example: Does this NFA accept the string 1100? YES

$q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_2 \xrightarrow{0} q_3$ (accept)

$q_0 \xrightarrow{1} q_1 \xrightarrow{1} \perp$ (fail)

$q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_1 \xrightarrow{0} q_2$ (reject)

Nondeterminism

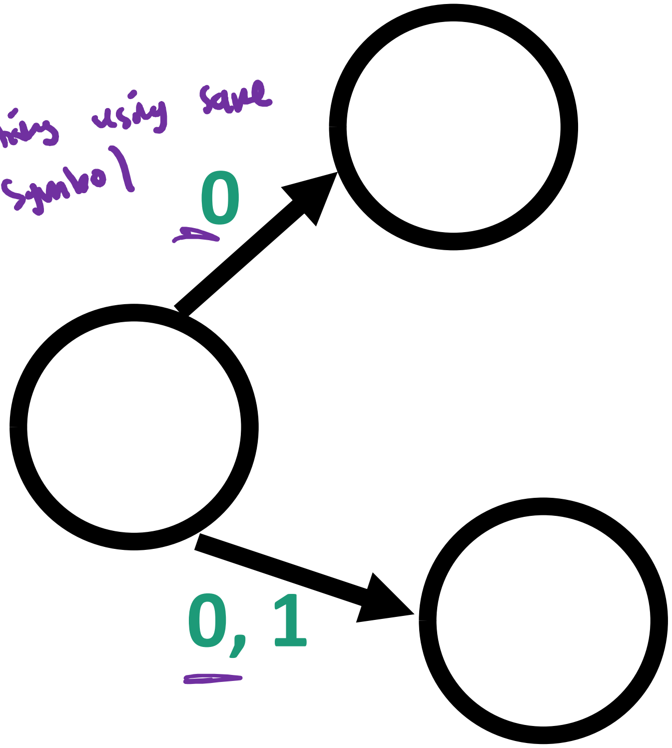


Example: Does this NFA accept the string 11?

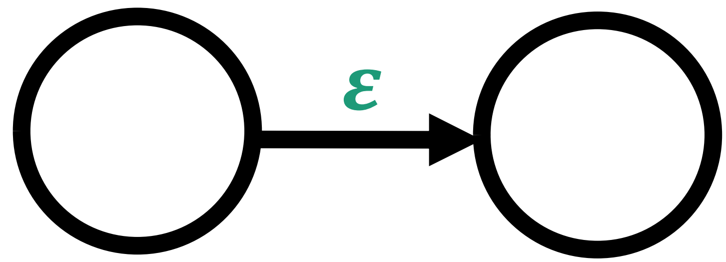
NO Every computation path leads to rejection or failure

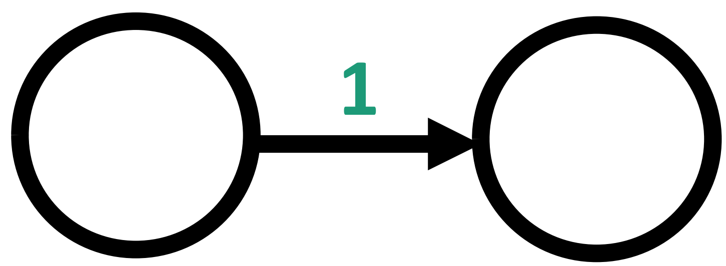
Some special transitions

Multiple transitions using same symbol



Transition w/o reading a symbol





No transition following 0 symbol