BU CS 332 – Theory of Computation

https://forms.gle/EmwazuipdvDh21yLA

Lecture 5:

- Closure Properties
- Regular Expressions

Reading:

Sipser Ch 1.2-1.3

Mark Bun

February 5, 2024

Last Time

- Nondeterministic Finite Automata
- NFAs vs. DFAs
  - Subset construction: NFA $\rightarrow$ DFA

Closure Properties

An Analogy

In algebra, we try to identify operations which are common to many different mathematical structures

Example: The integers $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ are **closed** under

- Addition: $x + y$
- Multiplication: $x \times y$
- Negation: $-x$
- …but NOT Division: $x \,/\, y$

We'd like to investigate similar closure properties of the class of regular languages

Regular operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Union: $A \cup B = \{w \mid w \in A \textbf{ or } w \in B\}$

Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$

Star: $A^* =$

Other operations

Let $A, B \subseteq \Sigma^*$ be languages. Define

Complement: $\bar{A} = \{w \mid w \notin A\}$

Intersection: $A \cap B = \{w \mid w \in A \textbf{ and } w \in B\}$

Reverse: $A^R = \{w \mid w^R \in A\}$

Operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Regular Operations

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$
- Star: $A^* = \{ w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in A\}$

Complement: $\bar{A} = \{x \mid x \notin A\}$

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Reverse: $A^R = \{ a_1 a_2 \ldots a_n \mid a_n \ldots a_1 \in A\}$

Theorem: The class of regular languages is closed under all six of these operations, i.e., if A and B are regular, applying any of these operations yields a regular language

Proving Closure Properties

Complement

Complement: $\bar{A} = \{ w \mid w \notin A\}$

Theorem: If A is regular, then $\bar{A}$ is also regular

Proof idea:

Complement, Formally

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing a language A. Which of the following represents a DFA recognizing $\bar{A}$?

a) $(F, \Sigma, \delta, q_0, Q)$

b) $(Q, \Sigma, \delta, q_0, Q \setminus F)$, where $Q \setminus F$ is the set of states in Q that are not in F

c) $(Q, \Sigma, \delta', q_0, F)$ where $\delta'(q, s) = p$ such that $\delta(p, s) = q$

d) None of the above

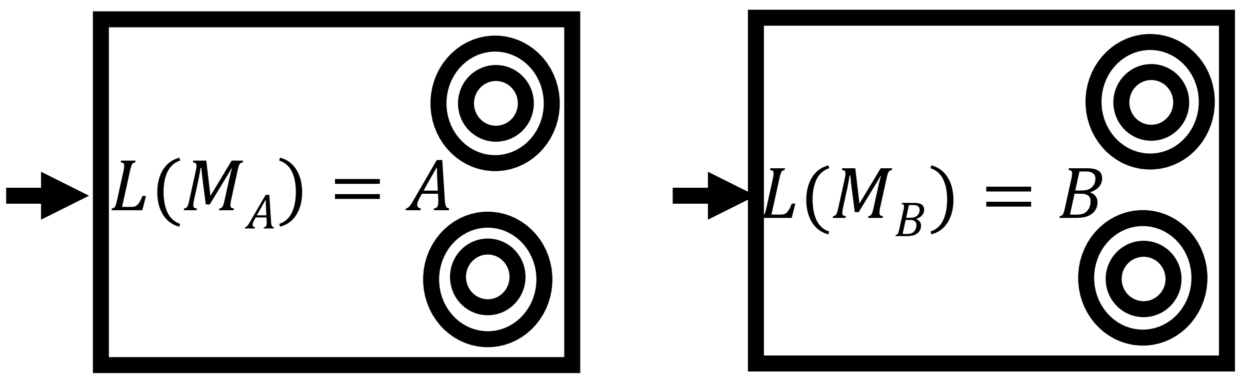
Closure under Concatenation

Concatenation: $A \circ B = \{ xy \mid x \in A, y \in B \}$

Theorem. If A and B are regular, then $A \circ B$ is also regular.

Proof idea: Given DFAs M_A and M_B, construct NFA by

- Connecting all accept states in M_A to the start state in M_B.
- Make all states in M_A non-accepting.



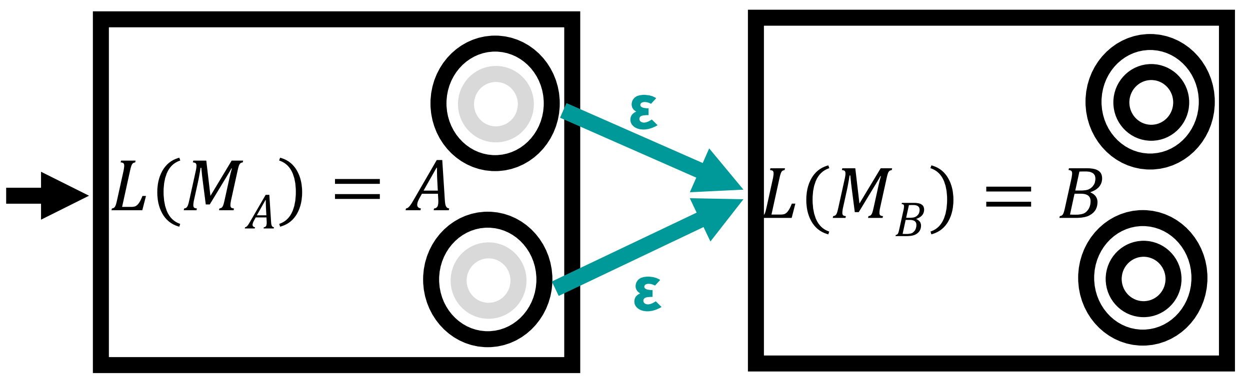
Closure under Concatenation

Concatenation: $A \circ B \ = \ \{ xy \ | x \in A, y \in B \}$

Theorem. If A and B are regular, then $A \circ B$ is also regular.

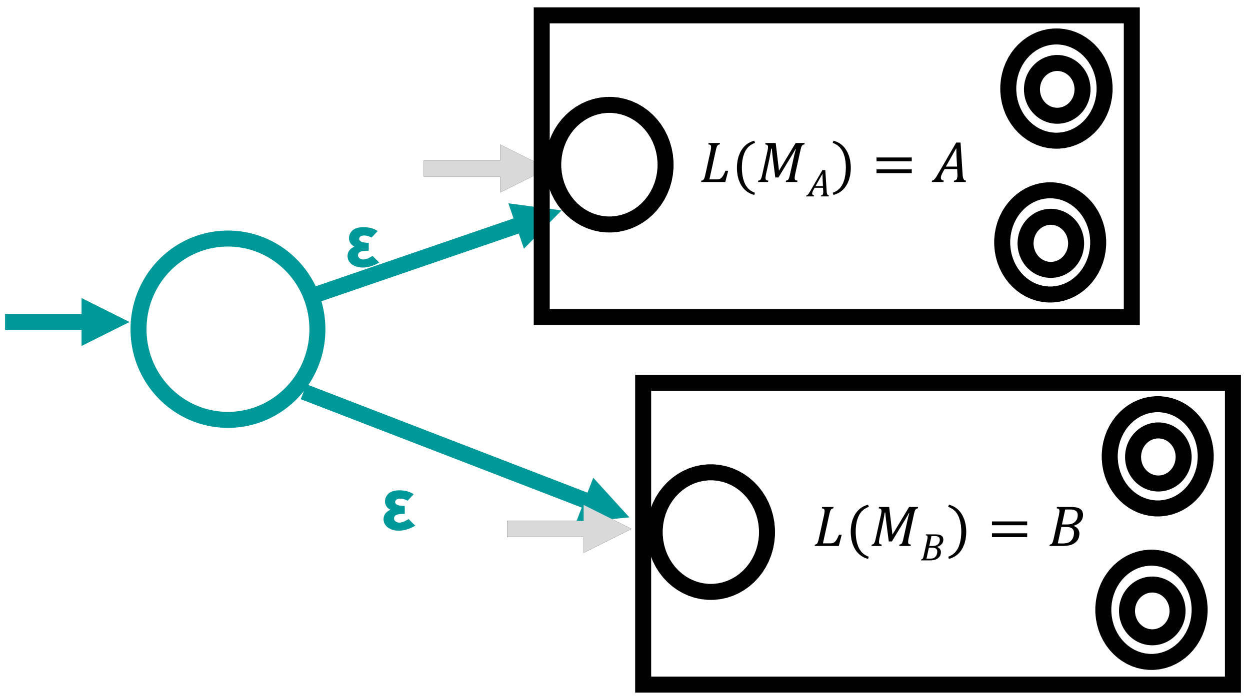
Proof idea: Given DFAs M_A and M_B, construct NFA by

- Connecting all accept states in M_A to the start state in M_B.
- Make all states in M_A non-accepting.



A Mystery Construction

Given DFAs M_A recognizing A and M_B recognizing B, what does the following NFA recognize?

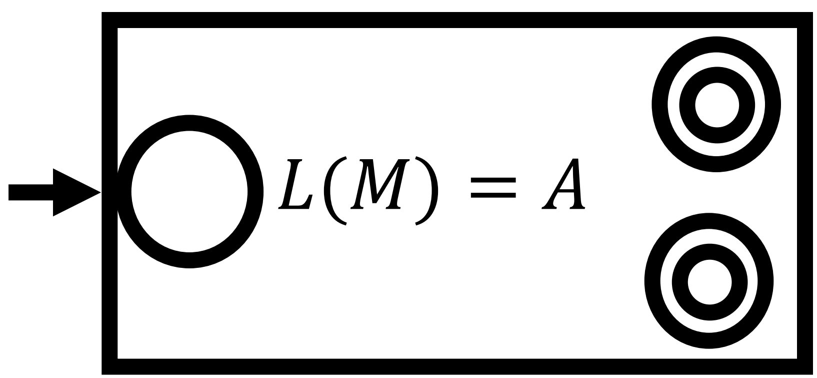


a) $A \cup B$
b) $A \circ B$
c) $A \cap B$
d) $\{\varepsilon\} \cup A \cup B$

Closure under Star

Star: $A^* = \{\, a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A\}$

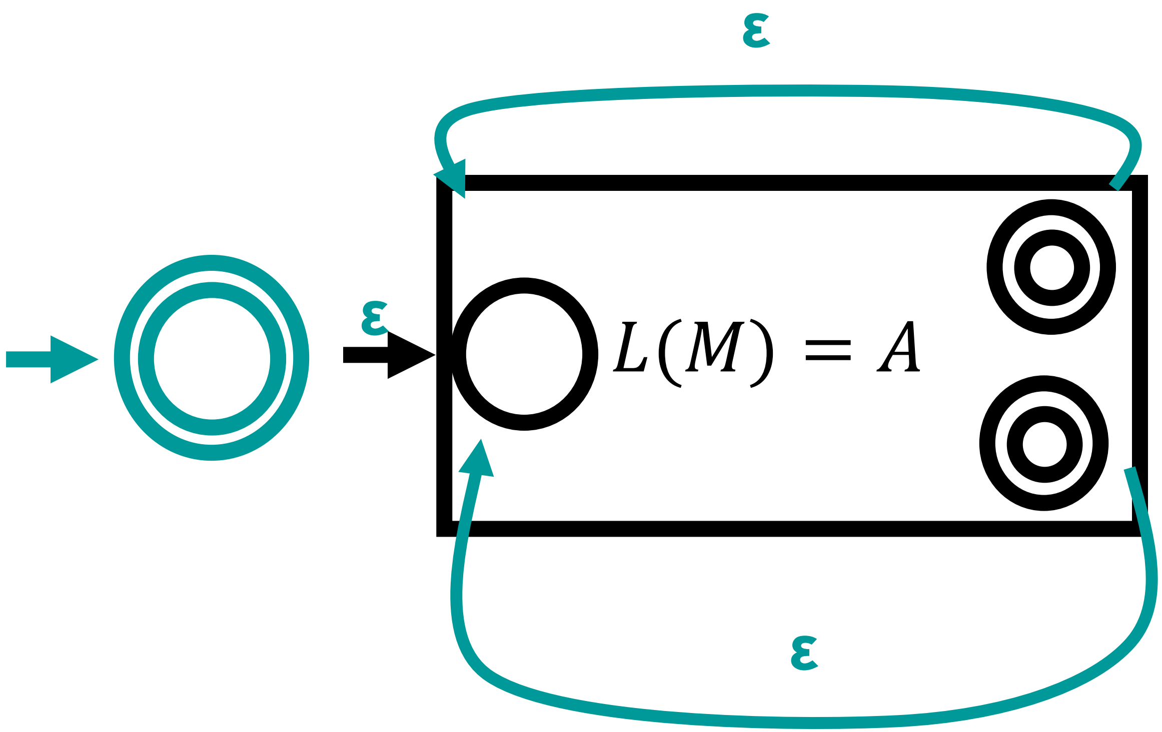
Theorem. If A is regular, then A^* is also regular.



Closure under Star

Star: $A^* = \{ a_1 a_2 ... a_n \,|\, n \geq 0 \text{ and } a_i \in A\}$

Theorem. If A is regular, then A^* is also regular.



On proving your own closure properties

You'll have homework/test problems of the form "show that the regular languages are closed under some operation"

What would Sipser do?

- Give the "proof idea": Explain how to take machine(s) recognizing regular language(s) and create a new machine
- Explain in a few sentences why the construction works
- Give a formal description of the construction
- No need to formally prove that the construction works

Regular Expressions

Regular Expressions

- A different way of describing regular languages
- A regular expression expresses a (possibly complex) language by combining simple languages using the regular operations

"Simple" languages: $\emptyset, \{\varepsilon\}, \{a\}$ for some $a \in \Sigma$

Regular operations:

Union: $A \cup B$

Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0$ and $a_i \in A\}$

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are

   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$)

$(a \circ b)$ $\quad$ $((((a \circ (b^*)) \circ c) \cup (((a^*) \circ b))^*))$ $\quad$ $(\emptyset^*)$

Regular Expressions – Semantics

$L(R) =$ the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\left((R_1^*)\right) = (L(R_1))^*$

Regular Expressions – Example

$L(((a^*) \circ (b^*))) =$

a) $\{a^n b^n \mid n \geq 0\}$

b) $\{a^m b^n \mid m, n \geq 0\}$

c) $\{(ab)^n \mid n \geq 0\}$

d) $\{a, b\}^*$

Simplifying Notation

- Omit $\circ$ symbol: $(ab) = (a \circ b)$

- Omit many parentheses, since union and concatenation are associative:

$$(a \cup b \cup c) = (a \cup (b \cup c)) = ((a \cup b) \cup c)$$

- Order of operations: Evaluate star, then concatenation, then union

$$ab^* \cup c = (a(b^*)) \cup c$$

Examples

Let $\Sigma = \{0, 1\}$

1. $\{w \mid w \text{ contains exactly one } 1\}$

2. $\{w \mid w \text{ has length at least 3 and its third symbol is } 0\}$

3. $\{w \mid \text{every odd position of } w \text{ is } 1\}$

Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$

- For regex R, the regex $R^{+} = RR^{*}$

Regexes in the Real World

`grep` = globally search for a regular expression and print matching lines

```
$ grep '^xy*z' myfile
xyz
xyzde
xzz
xz
xyyz
xyyyz
xyyyyz
$ grep '^x.*z' myfile
xyz
xyzde
xxz
xzz
x\z
x*z
xz
x z
xYz
xyyz
xyyyz
xyyyyz
$ grep '^x\*z' myfile
x*z
$ grep '\\' myfile
x\z
$
```

Equivalence of Regular Expressions, NFAs, and DFAs

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Base cases:

$$R = \emptyset$$

$$R = \varepsilon$$

$$R = a$$

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

What should the inductive hypothesis be?

a) Suppose **some** regular expression of length k can be converted to an NFA
b) Suppose **every** regular expression of length k can be converted to an NFA
c) Suppose **every** regular expression of length **at most** k can be converted to an NFA
d) None of the above

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Inductive step:

$$R = (R_1 \cup R_2)$$

$$R = (R_1 R_2)$$

$$R = (R_1^*)$$

Example

Convert $(1(0 \cup 1))^*$ to an NFA