# BU CS 332 – Theory of Computation

https://forms.gle/5sTNDCU1QtEemHHM7

## Lecture 6:

- Regexes = NFAs
- Limitations of Finite Automata

Reading:

Sipser Ch 1.3

"Myhill-Nerode" note

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# Regular Expressions – Syntax

A regular expression $R$ is defined recursively using the following rules:

1. $\varepsilon$, $\emptyset$, and $a$ are regular expressions for every $a \in \Sigma$

2. If $R_1$ and $R_2$ are regular expressions, then so are
   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and $(R_1^*)$

Examples: (over $\Sigma = \{a, b, c\}$) (with simplified notation)

$ab$ $ab^*c \cup (a^*b)^*$ $\emptyset$

# Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\left((R_1^*)\right) = (L(R_1))^*$

Example: $L(a^*b^*) = \{a^m b^n \mid m, n \geq 0\}$

# Syntactic Sugar

- For alphabet $\Sigma$, the regex $\Sigma$ represents $L(\Sigma) = \Sigma$

- For regex $R$, the regex $R^{+} = RR^{*}$

# Regexes in the Real World

`grep` = globally search for a regular expression and print matching lines

```
$ grep '^xy*z' myfile
xyz
xyzde
xzz
xz
xyyz
xyyyz
xyyyyz
$ grep '^x.*z' myfile
xyz
xyzde
xxz
xzz
x\z
x*z
xz
x z
xYz
xyyz
xyyyz
xyyyyz
$ grep '^x\*z' myfile
x*z
$ grep '\\' myfile
x\z
$ 
```

# Regular Expressions Describe Regular Languages

Theorem: A language $A$ is regular if and only if it is described by a regular expression

Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

# Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Base cases:

$$R = \emptyset$$

$$R = \varepsilon$$

$$R = a$$

# Regular expression -> NFA

**Theorem 1:** Every regex has an equivalent NFA

Proof: Induction on size of a regex

What should the inductive hypothesis be?

a) Suppose **some** regular expression of length $k$ can be converted to an NFA
b) Suppose **every** regular expression of length $k$ can be converted to an NFA
c) Suppose **every** regular expression of length **at most** $k$ can be converted to an NFA
d) None of the above

# Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Inductive step:

$$R = (R_1 \cup R_2)$$

$$R = (R_1 R_2)$$

$$R = (R_1^*)$$

# Example

Convert $(1(0 \cup 1))^*$ to an NFA

# Regular Expressions Describe Regular Languages

**Theorem:** A language $A$ is regular if and only if it is described by a regular expression

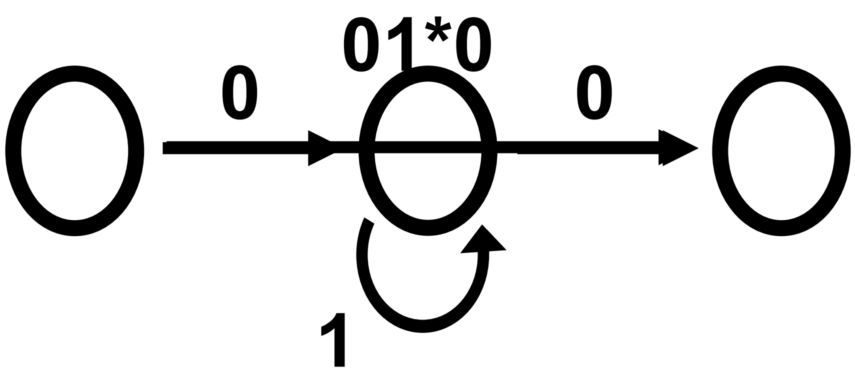
**Theorem 1:** Every regular expression has an equivalent NFA

**Theorem 2:** Every NFA has an equivalent regular expression

# NFA -> Regular expression

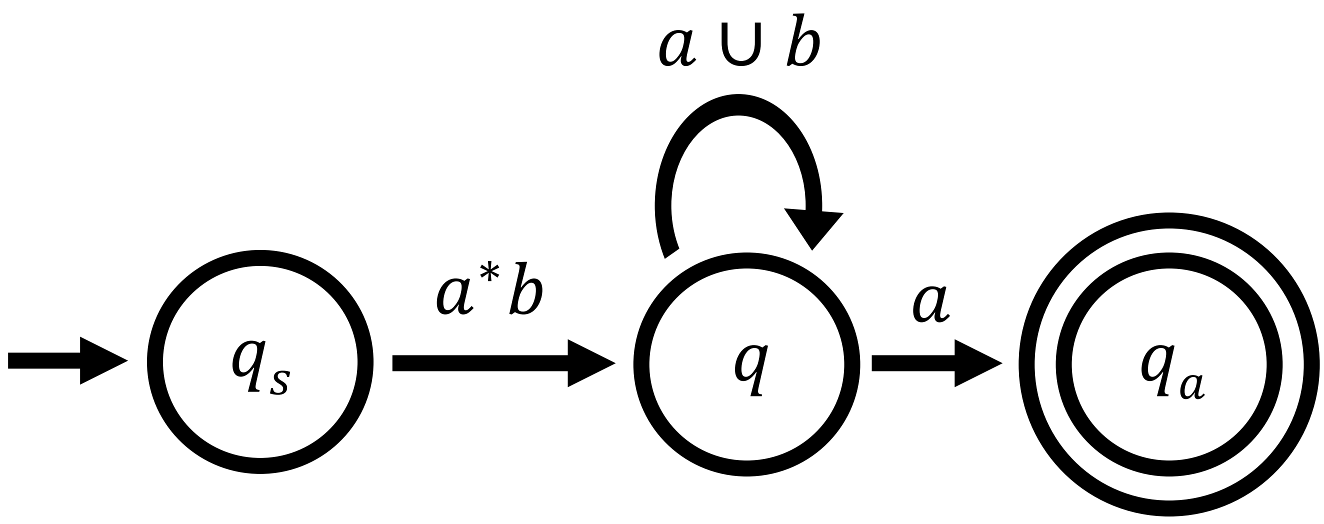
Theorem 2: Every NFA has an equivalent regex

Proof idea: Simplify NFA by “ripping out” states one at a time and replacing with regexes

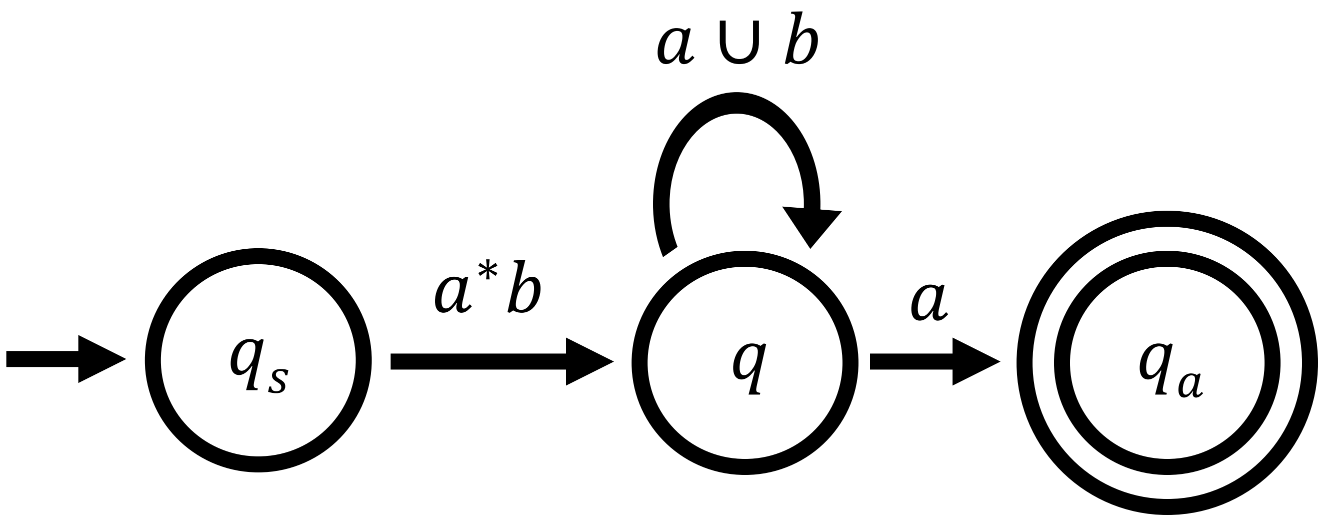


# Generalized NFAs

- **Every transition is labeled by a regex**
- One start state with only outgoing transitions
- Only one accept state with only incoming transitions
- Start state and accept state are distinct



# Generalized NFA Example



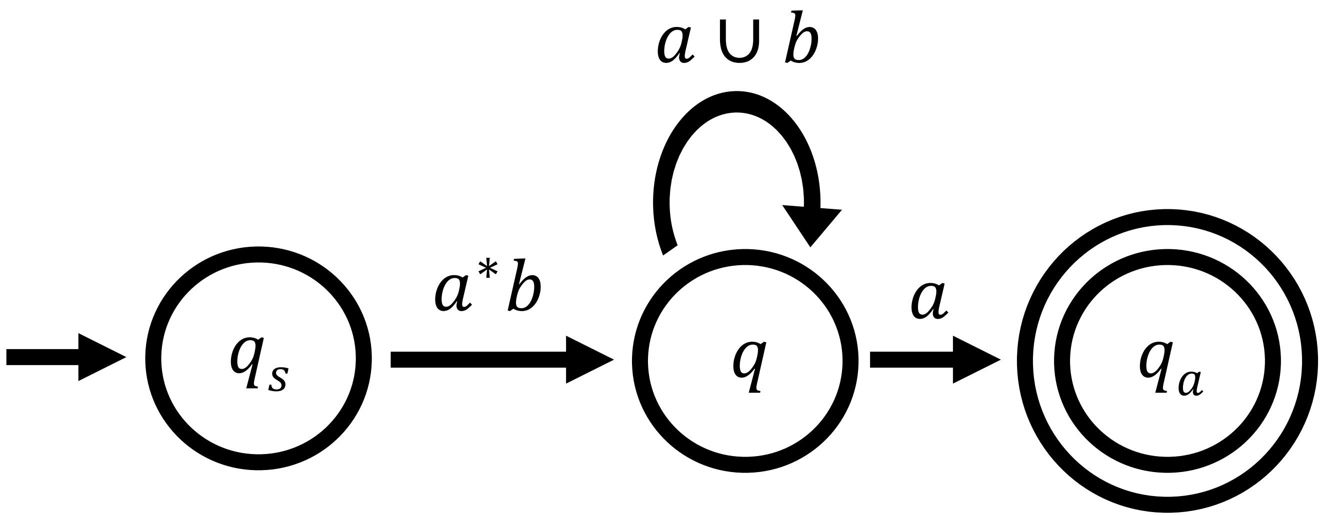

$$R(q_s, q) =$$

$$R(q_a, q) =$$

$$R(q, q_s) =$$

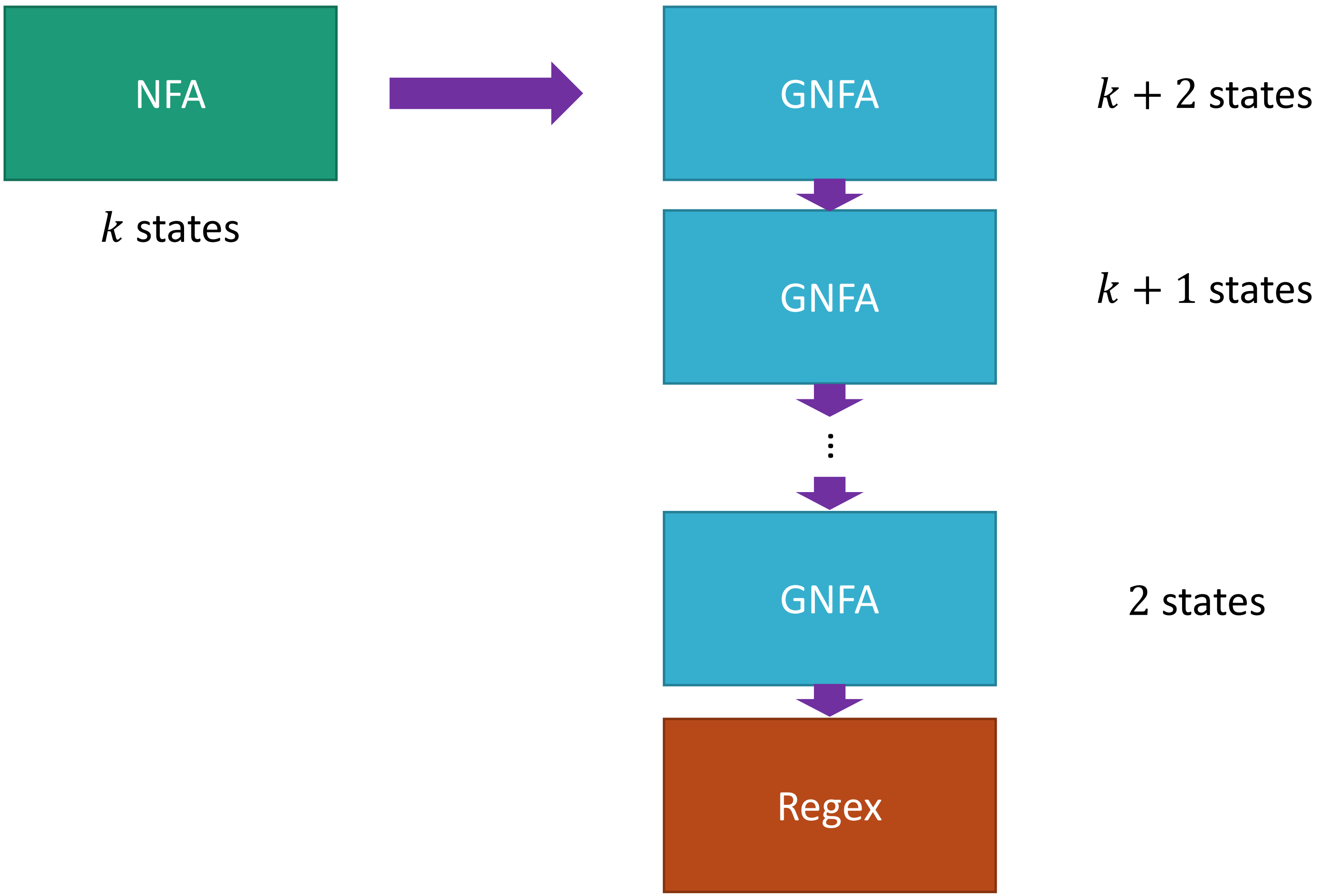
# Which of these strings is accepted?

Which of the following strings is accepted by this GNFA?

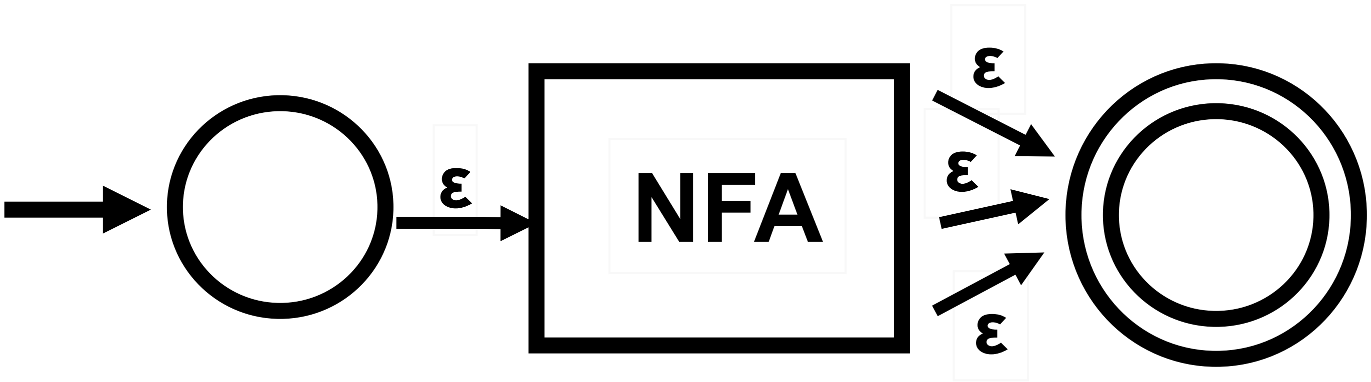



a) $aaa$
b) $aabb$
c) $bbb$
d) $bba$

# NFA -> Regular expression



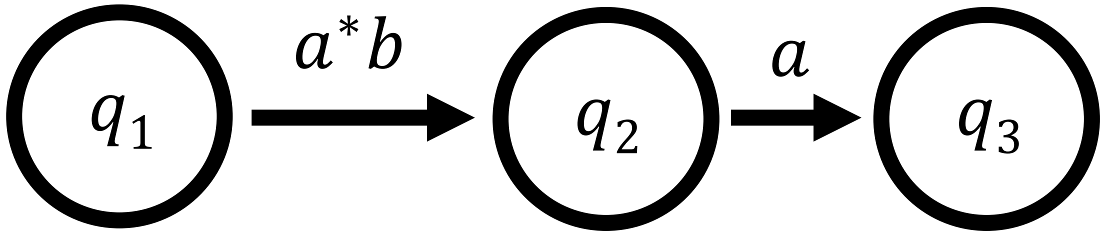
# NFA -> GNFA

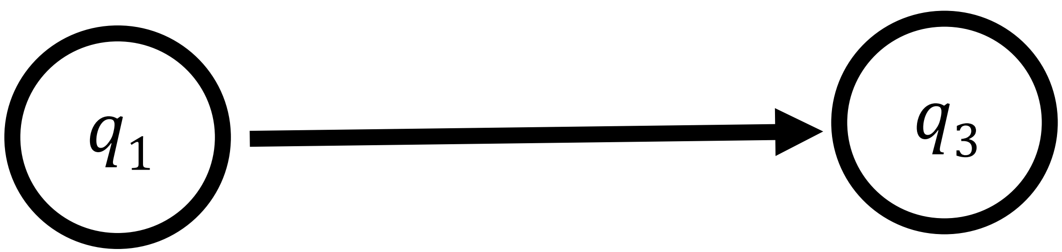



- Add a new start state with no incoming arrows.
- Make a unique accept state with no outgoing arrows.

# GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state

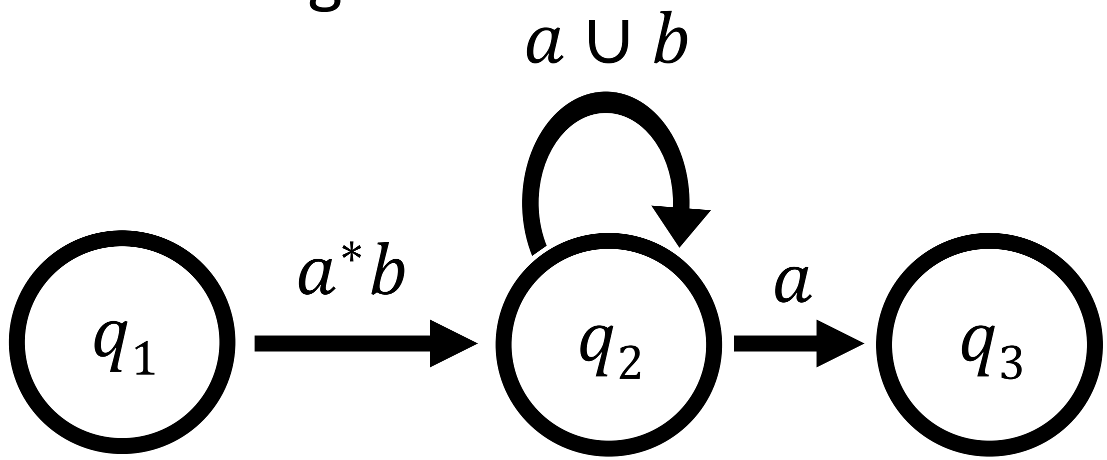


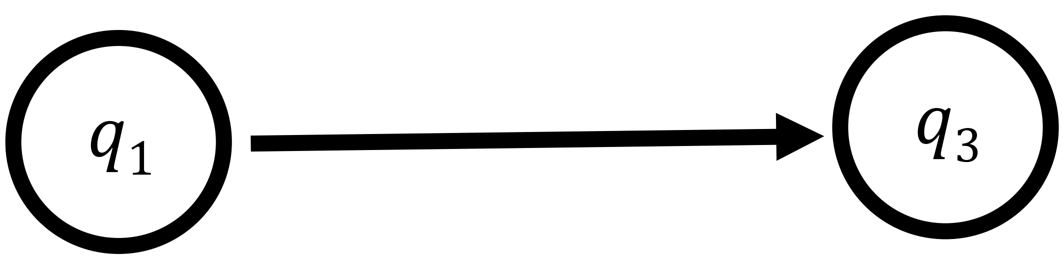



# GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state

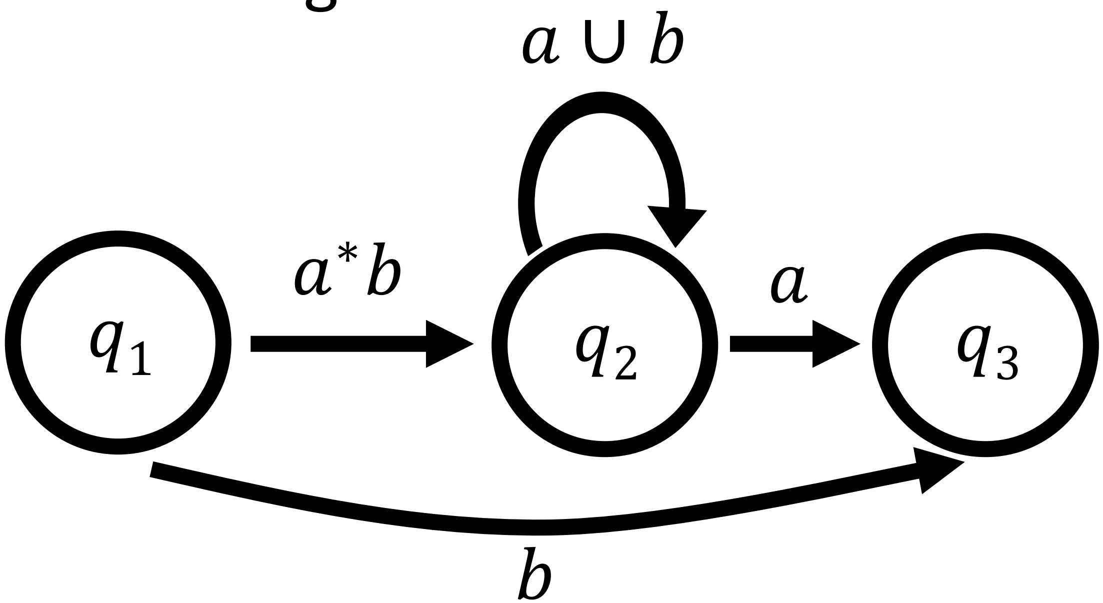
a) $a^*b(a \cup b)a$
b) $a^*b(a \cup b)^*a$
c) $a^*b \cup (a \cup b) \cup a$
d) None of the above

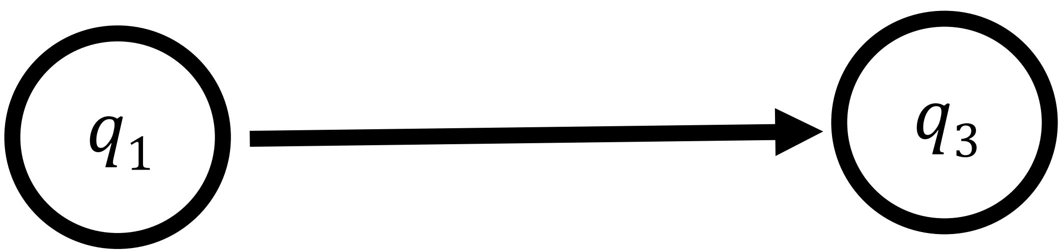


# GNFA -> Regular expression

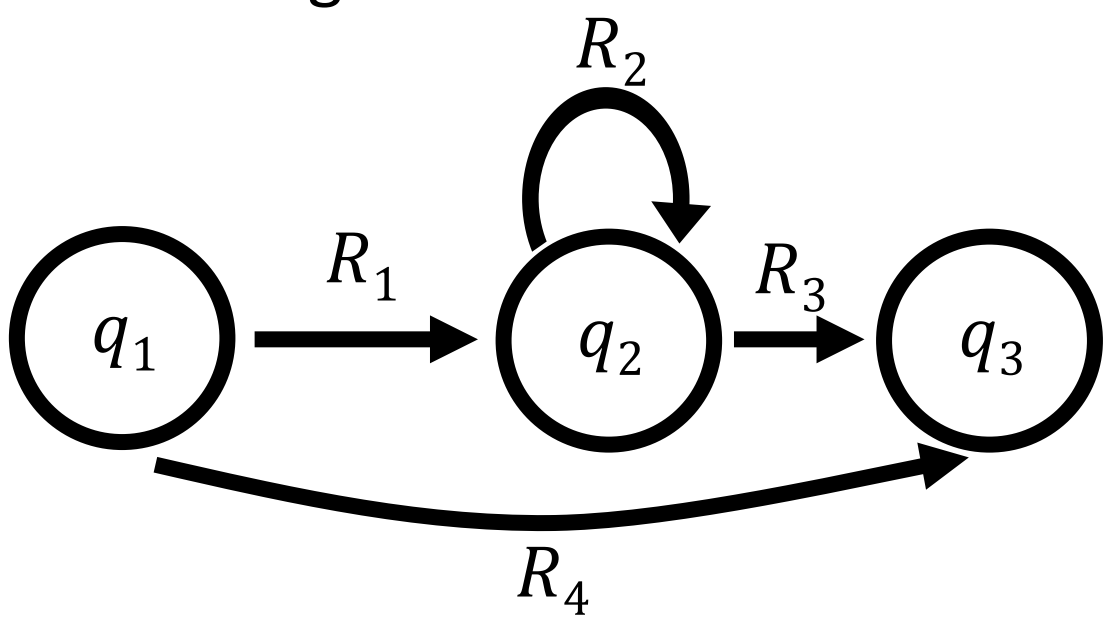
Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state

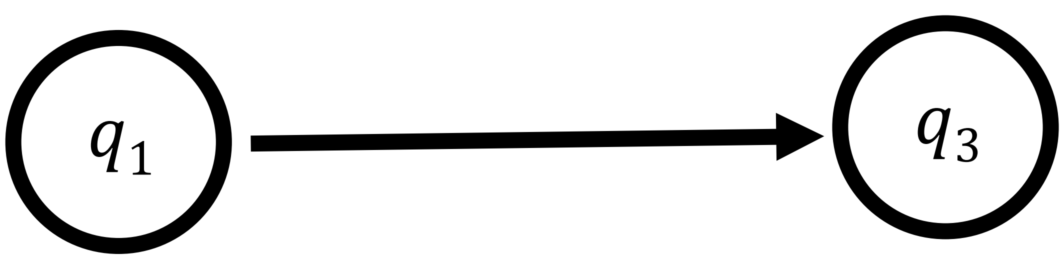


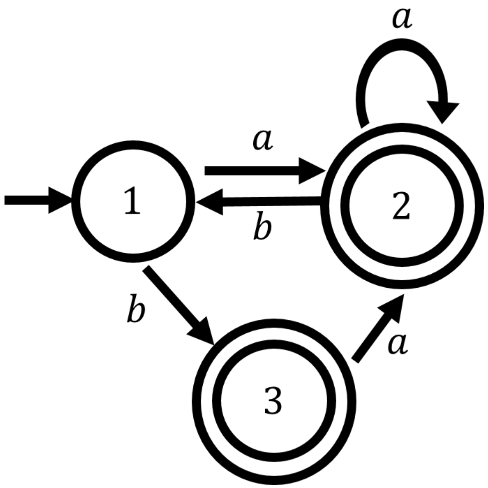



# GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state




a
a
1
2
b
b
3
a

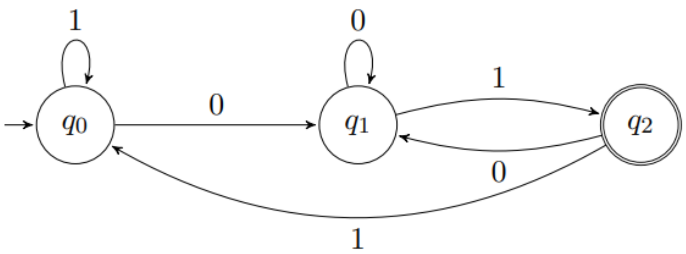
# Limitations of Finite Automata

# Motivating Questions

- We’ve seen techniques for showing that languages are regular

- How can we tell if we’ve found the smallest DFA recognizing a language?
- Are all languages regular? How can we prove that a language is not regular?

# An Example




$A = \{\, w \in \{0,1\}^* \mid w \text{ ends with } 01 \,\}$

Claim: *Every* DFA recognizing $A$ needs at least 3 states

Proof: Let $M$ be any DFA recognizing $A$. Consider running $M$ on each of $x = \varepsilon, y = 0, w = 01$

# A General Technique

$A = \{ w \in \{0,1\}^* \mid w \text{ ends with } 01 \}$

**Definition:** Strings $x$ and $y$ are **distinguishable** by $L$ if there exists a "distinguishing extension" $z \in \Sigma^*$ such that exactly one of $xz$ or $yz$ is in $L$.

Ex. $x = \varepsilon, \quad y = 0$

**Definition:** A set of strings $S$ is **pairwise distinguishable** by $L$ if every pair of distinct strings $x, y \in S$ is distinguishable by $L$.

Ex. $S = \{\varepsilon, 0, 01\}$

# A General Technique

Theorem: If $S$ is pairwise distinguishable by $L$, then every DFA recognizing $L$ needs at least $|S|$ states

Proof: Let $M$ be a DFA with $< |S|$ states. By the pigeonhole principle, there are $x, y \in S$ such that $M$ ends up in same state on $x$ and $y$

# Back to Our Example

$A = \{\, w \in \{0,1\}^* \mid w \text{ ends with } 01 \,\}$

Theorem: If $S$ is pairwise distinguishable by $L$, then every DFA recognizing $L$ needs at least $|S|$ states

$S = \ \{\varepsilon, 0, 01\}$

# Another Example

$B = \{ w \in \{0,1\}^* \mid |w| = 2 \}$

Theorem: If $S$ is pairwise distinguishable by $L$, then every DFA recognizing $L$ needs at least $|S|$ states

$S = \{ \qquad \}$

# Distinguishing Extension

Which of the following is a distinguishing extension for $x = 0$ and $y = 00$ for language $B = \{\, w \in \{0, 1\}^* \mid |w| = 2 \,\}$ ?

a) $z = \varepsilon$

b) $z = 0$

c) $z = 1$

d) $z = 00$