BU CS 332 – Theory of Computation

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Lecture 14:

- Undecidability
- Reductions

Reading:

Sipser Ch 4.2, 5.1

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Where we are and where we're going

Church-Turing thesis: TMs capture all algorithms

Consequence: studying the limits of TMs reveals the limits of computation

Last time: Countability, uncountability, and diagonalization

Today: Existential proof that there are undecidable and unrecognizable languages

An explicit undecidable language

Reductions: Relate decidability / undecidability of different problems

Last time: A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

$= \{S \mid S \subseteq X\}$

Proof: Assume for the sake of contradiction that there is a surjection $f: X \to P(X)$

Goal: Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$

Diagonalization argument

Row corresponds to $f(x_1)$ which is some subset of X

Assume a surjection $f: X \to P(X)$

Is $x_1 \in f(x_1)$?

Is $x_2 \in f(x_1)$?

x	$x_1 \in f(x)$?	$x_2 \in f(x)$?	$x_3 \in f(x)$?	$x_4 \in f(x)$?	…
x_1	~~Y~~ N	N	Y	Y	
x_2	N	~~N~~ Y	Y	Y	
x_3	Y	Y	~~Y~~ N	N	
x_4	N	N	Y	N	
⋮					⋱

$x_1 \notin S$ $\quad$ $x_2 \in S$ $\quad$ $x_3 \notin S$

Define S by flipping the diagonal:

Put $\quad x_i \in S \quad \Longleftrightarrow \quad x_i \notin f(x_i)$

Last time: A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a surjection $f: X \rightarrow P(X)$

$x \mapsto S$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$

$y \in S \Rightarrow y \notin S$ ✗

$y \notin S \Rightarrow y \in S$ ✗

f can't possibly be a surjection

Undecidable Languages

Undecidability / Unrecognizability

Definition: A language L is **undecidable** if there is no TM deciding L

Definition: A language L is **unrecognizable** if there is no TM recognizing L

An existential proof

Theorem: There exists an undecidable language over $\{0,1\}$

Proof: $\exists L \subseteq \{0,1\}^*$ s.t. L is not decidable

- Set of all encodings of TM deciders: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $\{L \mid L \subseteq \{0,1\}^*\}$

a) $\{0,1\}$

b) $\{0,1\}^*$

c) $P(\{0,1\}^*)$: The set of all subsets of $\{0,1\}^*$

d) $P(P(\{0,1\}^*))$: The set of all subsets of the set of all subsets of $\{0,1\}^*$

An existential proof

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0, 1\}^*$

Set of all languages over $\{0, 1\}$: $P(\{0, 1\}^*)$

There are more languages than there are TM deciders!

$\Rightarrow$ There must be an undecidable language

An existential proof

Theorem: There exists an unrecognizable language over $\{0, 1\}$

Proof:

$\{$TM deciders$\} \subsetneq \{$TM recognizers$\}$

Set of all encodings of TMs: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $P(\{0,1\}^*)$

$|\{$TM deciders$\}| = |\{$TM recognizers$\}| = |\mathbb{N}|$

There are more languages than there are TM recognizers!

$\Rightarrow$ There must be an unrecognizable language

“Almost all” languages are undecidable

But how do we actually find one?

An Explicit Undecidable Language

Our power set size proof

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

1) Assume, for the sake of contradiction, that there is a surjection $f: X \to P(X)$

2) "Flip the diagonal" to construct a set $S \in P(X)$ such that $f(x) \neq S$ for every $x \in X$

3) Conclude that f is not onto, a contradiction

Specializing the proof

Theorem: Let X be the set of all TM deciders. Then there exists an undecidable language in $P(\{0,1\}^*)$

$= \{L \mid L \subseteq \{0,1\}^*\}$

1) Assume, for the sake of contradiction, that $L: X \to P(\{0,1\}^*)$ is a surjection $L(M)$ = language recognized by M

2) "Flip the diagonal" to construct a language $UD \in P(\{0,1\}^*)$ such that $L(M) \neq UD$ for every $M \in X$

3) Conclude that L is not onto, a contradiction

An explicit undecidable language

TM M					
M_1					
M_2					
M_3					
M_4					
$\vdots$					

Why is it possible to enumerate all TMs like this?

a) The set of all TMs is finite

b) The set of all TMs is countably infinite

c) The set of all TMs is uncountable

An explicit undecidable language

Run TM M_1 on input $\langle M_1 \rangle$. If accept, Y
If rejects or loops, N

Run TM M_1 on input $\langle M_2 \rangle$

TM M	$M(\langle M_1\rangle)$?	$M(\langle M_2\rangle)$?	$M(\langle M_3\rangle)$?	$M(\langle M_4\rangle)$?		$D(\langle D\rangle)$?
M_1	~~Y~~ N	N	Y	Y	…	
M_2	N	~~N~~ Y	Y	Y		
M_3	Y	Y	Y	N		
M_4	N	N	Y	N		
⋮					⋱	
D						~~Y~~ N ~~N~~ Y

$UD = \{\langle M\rangle \mid M$ is a TM that does not accept on input $\langle M\rangle\}$

Claim: UD is undecidable

If M_1 accepts on input $\langle M_1 \rangle$, then $\langle M_1 \rangle \notin UD$
If M_2 does not accept on $\langle M_2 \rangle$ then $\langle M_2 \rangle \in UD$ . . .

def. of UD

If some TM D were to decide UD
1) D accepts when run on $\langle D \rangle$ $\Rightarrow \langle D \rangle \notin UD \Rightarrow$ D messed up on $\langle D \rangle$ $\not\leftarrow$
2) D does not accept when run on $\langle D \rangle \Rightarrow \langle D \rangle \in UD \Rightarrow$ D messed up on $\langle D \rangle$ $\leftarrow$

An explicit undecidable language

Theorem: $UD = \{\langle M \rangle \mid M$ is a TM that does not accept on input $\langle M \rangle\}$ is undecidable

Proof: Suppose for contradiction that TM D decides UD

Examine two cases:

1) D accepts on input <D>
   ⇒ <D> ∉ UD {definition of UD}
   ⇒ D accepts the string <D> which is not in UD, so D made a mistake, and hence does not decide UD ↯

2) D does not accept input <D>
   ⇒ <D> ∈ UD {definition of UD}
   ⇒ D fails to accept <D> which is in UD ↯.

A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

Proof: Assume for the sake of contradiction that TM H decides A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Idea: Show that H can be used to construct a decider for the (undecidable) language UD -- a contradiction.

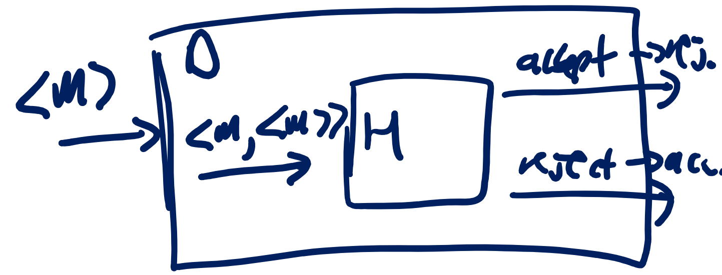
A more useful undecidable language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Proof (continued):

Suppose, for contradiction, that H decides A_{TM}

Consider the following TM D:



"On input $\langle M\rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. If H accepts, reject. If H rejects, accept."

Claim: D decides $UD = \{\langle M\rangle \mid \text{TM } M \text{ does not accept } \langle M\rangle\}$

Proof. 1) If $\langle M\rangle \in UD$, WTS D accepts on input $\langle M\rangle$

why? $\langle M\rangle \in UD \Rightarrow \langle M\rangle$ does not accept on input $\langle M\rangle$

$\Rightarrow \langle M, \langle M\rangle\rangle \notin A_{TM}$

$\Rightarrow H(\langle M, \langle M\rangle\rangle)$ reject $\Rightarrow D$ accepts ✓

2) If $\langle M\rangle \notin UD \Rightarrow \langle M\rangle$ accepts input $\langle M\rangle$

$\Rightarrow \langle M, \langle M\rangle\rangle \in A_{TM} \Rightarrow H(\langle M, \langle M\rangle\rangle)$ accepts

$\Rightarrow D$ rejects ✓

...but this language is undecidable!

Unrecognizable Languages

Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Corollary: $\overline{A_{\text{TM}}}$ is unrecognizable

Universal TM shows A_{TM} is recognizable.

A_{TM} is undecidable $\overset{\text{Thm}}{\Longrightarrow}$ Either ~~A_{TM} is unrecognizable~~ or

$\overline{A_{TM}}$ is unrecognizable

Proof of Theorem:

$\Rightarrow$) If L decidable, then L and $\overline{L}$ recognizable

L decidable $\Rightarrow$ L recognizable

$\hookrightarrow$ $\overline{L}$ is decidable [decidable langs. closed under complement]

$\Rightarrow$ $\overline{L}$ is recognizable

Unrecognizable Languages

Theorem: A language L is decidable if and only if L and $\bar{L}$ are both Turing-recognizable.

Proof continued:

$\Leftarrow$ If L recognizable and $\bar{L}$ recognizable, then L is decidable

Proof. $\exists$ TM M recognizing L and TM N recognizing $\bar{L}$.

The following TM decides L:

On input w:

1) Copy w to tapes 2 and 3

2) Do the following until a decision is made:

- Run M for one additional step on tape 2
  If accepts, accept
- Run N for one additional step on tape 3
  If accepts, reject.

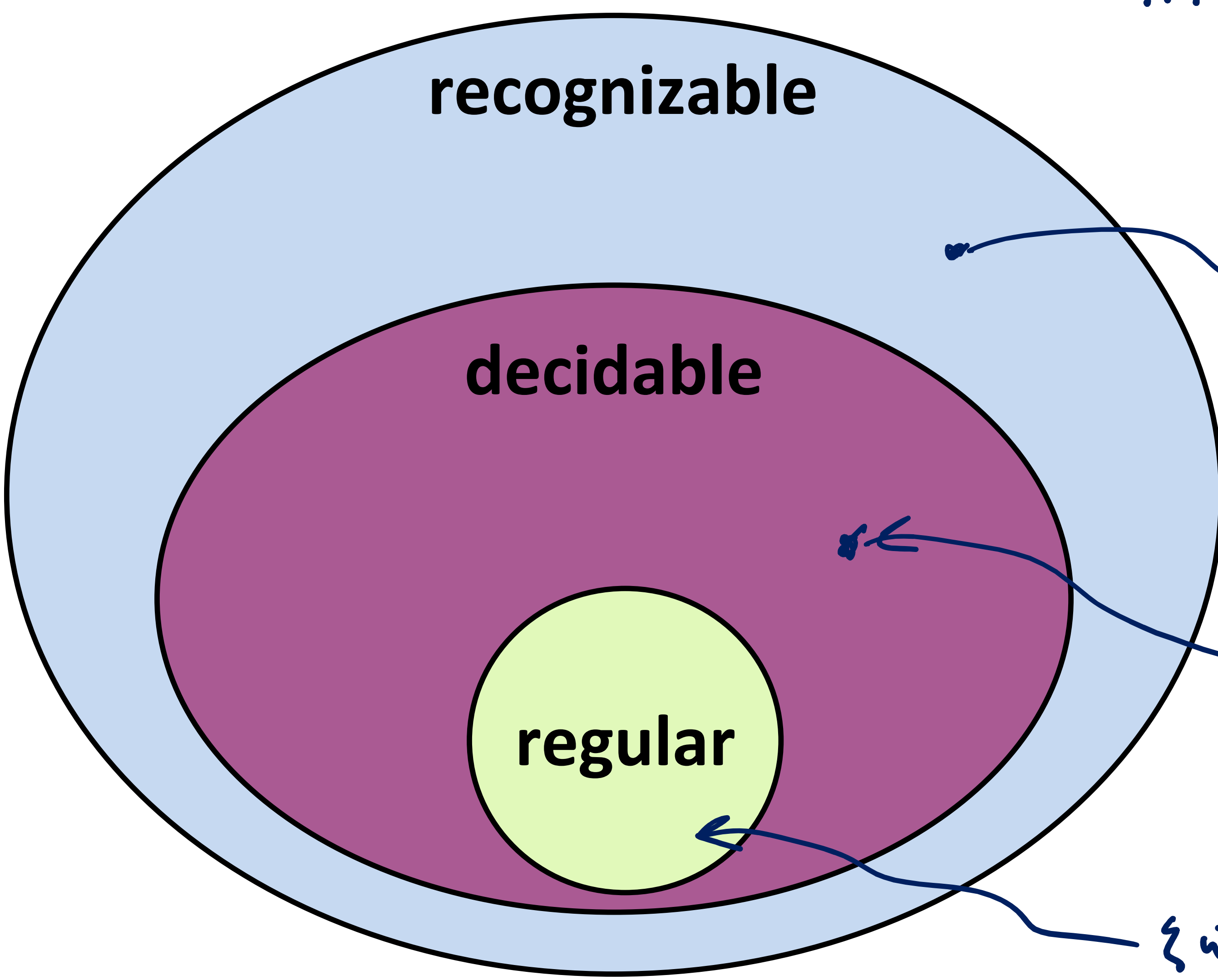
If $w \in L$:
Eventually M accepts w
so TM accepts

If $w \notin L$:
Eventually N accept w
so TM rejects.

Classes of Languages

$\overline{A_{TM}}$

Recognizable langs are not closed under complement



A_{TM}

$\{0^n 1^n \mid n \geq 0\}$

$\{w \mid w \text{ contains } 00\}$

Reductions

Scientists vs. Engineers

A computer scientist and an engineer are stranded on a desert island. They find two palm trees with one coconut on each. The engineer climbs a tree, picks a coconut and eats.

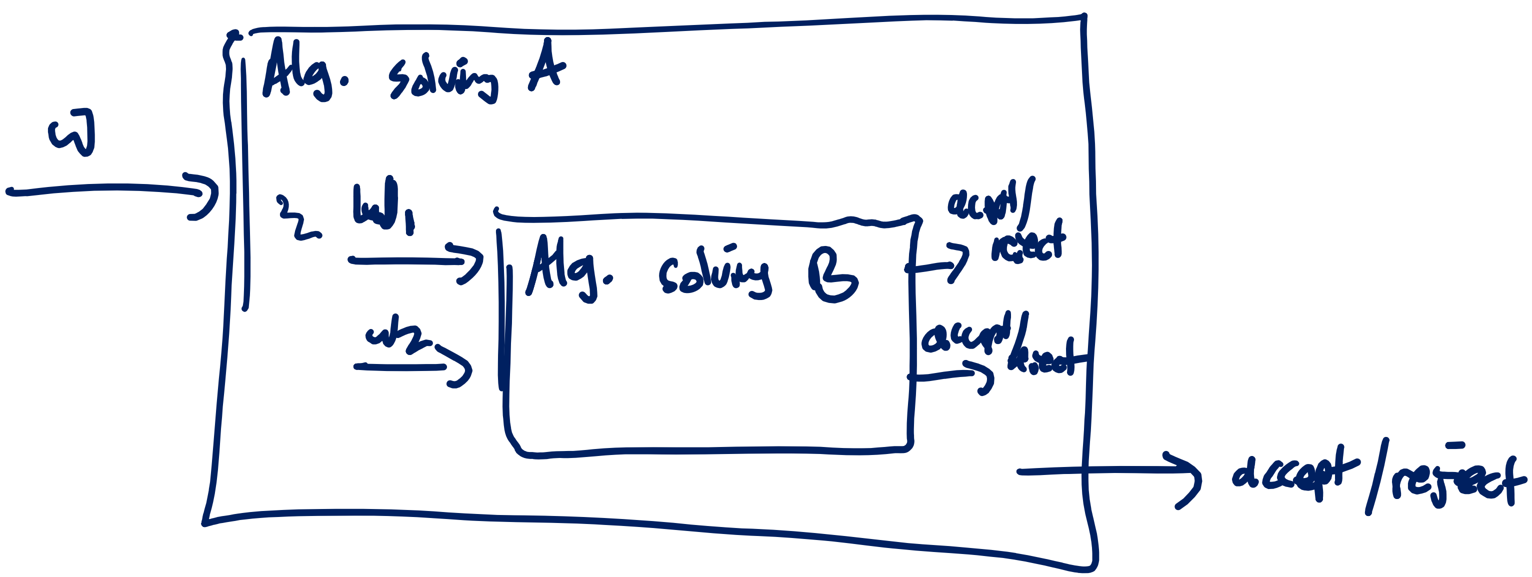
The computer scientist climbs the second tree, picks a coconut, climbs down, climbs up the first tree and places it there, declaring success.

“Now we’ve reduced the problem to one we’ve already solved.” (Please laugh)

Reductions

A reduction from problem A to problem B is an algorithm solving problem A which uses an algorithm solving problem B as a subroutine

If such a reduction exists, we say "A reduces to B"



Reductions

A reduction from problem A to problem B is an algorithm solving problem A which uses an algorithm solving problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

If A reduces to B, and B is decidable, what can we say about A?

a) A is decidable

b) A is undecidable

c) A might be either decidable or undecidable