BU CS 332 – Theory of Computation

https://forms.gle/dNF9ECAsFxeJ48dp8

Lecture 17:

- Mapping Reductions

Reading:

Sipser Ch 5.3

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Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

Positive uses: If A reduces to B and B is decidable, then A is also decidable

Ex. E_{DFA} is decidable $\Rightarrow$ EQ_{DFA} is decidable

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

Ex. E_{TM} is undecidable $\Rightarrow$ EQ_{TM} is undecidable

🚩🚩 Warning 🚩🚩

What's wrong with the following "proof"?

Bogus "Theorem": A_{TM} is not Turing-recognizable

Bogus "Proof": Let R be an alleged recognizer for A_{TM}. We construct a recognizer S for unrecognizable language $\overline{A_{\text{TM}}}$:

On input $\langle M, w \rangle$:

1. Run R on input $\langle M, w \rangle$
2. If R accepts, reject. If R rejects, accept.

This sure looks like a reduction from $\overline{A_{\text{TM}}}$ to A_{TM}

Mapping Reductions: Motivation

1. How do we formalize the notion of a reduction?
2. How do we use reductions to show that languages are unrecognizable?
3. How do we protect ourselves from accidentally “proving” bogus statements about recognizability?

Computable Functions

Definition:

A function $f: \Sigma^* \rightarrow \Sigma^*$ is computable if there is a TM M which, given as input any $w \in \Sigma^*$, halts with only $f(w)$ on its tape. (“Outputs $f(w)$”)

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Example 1: $f(w) = \text{sort}(w)$

Example 2: $f(\langle x, y \rangle) = x + y$

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Example 3: $f(\langle M, w \rangle) = \langle M' \rangle$ where M is a TM, w is a string, and M' is a TM that ignores its input and simulates running M on w

Mapping Reductions

Definition:

Let $A, B \subseteq \Sigma^*$ be languages. We say A is mapping reducible to B, written

$$A \leq_{\mathrm{m}} B$$

if there is a computable function $f: \Sigma^* \to \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \Leftrightarrow f(w) \in B$

Mapping Reductions

Definition:

Language A is mapping reducible to language B, written

$$A \leq_{\mathrm{m}} B$$

if there is a computable function $f: \Sigma^* \to \Sigma^*$ such that for all strings $w \in \Sigma^*$, we have $w \in A \Leftrightarrow f(w) \in B$

If $A \leq_{\mathrm{m}} B$, which of the following is true?

a) $\bar{A} \leq_{\mathrm{m}} B$

b) $A \leq_{\mathrm{m}} \bar{B}$

c) $\bar{A} \leq_{\mathrm{m}} \bar{B}$

d) $\bar{B} \leq_{\mathrm{m}} \bar{A}$

Decidability

Theorem: If $A \leq_{\mathrm{m}} B$ and B is decidable, then A is also decidable

Proof: Let M be a decider for B and let $f: \Sigma^* \rightarrow \Sigma^*$ be a mapping reduction from A to B. We can construct a decider N for A as follows:

On input w:

1. Compute $f(w)$
2. Run M on input $f(w)$
3. If M accepts, accept.

   If it rejects, reject.

Undecidability

Theorem: If $A \leq_{\mathrm{m}} B$ and B is decidable, then A is also decidable

Corollary: If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is also undecidable

Old Proof: Equality Testing for TMs

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: EQ_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for EQ_{TM}. We construct a decider for E_{TM} as follows:

On input $\langle M\rangle$:

1. Construct TMs M_1, M_2 as follows:

   $M_1 = M$

   $M_2 =$ "On input x,
   1. Ignore x and reject"

2. Run R on input $\langle M_1, M_2\rangle$
3. If R accepts, accept. Otherwise, reject.

This is a reduction from E_{TM} to EQ_{TM}

New Proof: Equality Testing for TMs

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $E_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ (Hence EQ_{TM} is undecidable)

Proof: The following TM N computes the reduction f:

On input $\langle M \rangle$:

1. Construct TMs M_1, M_2 as follows:

   M_1 = M

   M_2 = "On input x,
   1. Ignore x and reject"

2. Output $\langle M_1, M_2 \rangle$

Mapping Reductions: Recognizability

Theorem: If $A \leq_{\mathrm{m}} B$ and B is recognizable, then A is also recognizable

Proof: Let M be a recognizer for B and let $f: \Sigma^* \rightarrow \Sigma^*$ be a mapping reduction from A to B. Construct a recognizer N for A as follows:

On input w:

1. Compute $f(w)$
2. Run M on input $f(w)$
3. If M accepts, accept.

   If it rejects, reject.

Unrecognizability

Theorem: If $A \leq_{\mathrm{m}} B$ and B is recognizable, then A is also recognizable

Corollary: If $A \leq_{\mathrm{m}} B$ and A is unrecognizable, then B is also unrecognizable

Corollary: If $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} B$, then B is unrecognizable

Recognizability and A_{TM}

Let L be a language. Which of the following is true?

a) If $L \leq_{\mathrm{m}} A_{\mathrm{TM}}$, then L is recognizable
b) If $A_{\mathrm{TM}} \leq_{\mathrm{m}} L$, then L is recognizable
c) If L is recognizable, then $L \leq_{\mathrm{m}} A_{\mathrm{TM}}$
d) If L is recognizable, then $A_{\mathrm{TM}} \leq_{\mathrm{m}} L$

Theorem: L is recognizable *if and only* if $L \leq_{\mathrm{m}} A_{\mathrm{TM}}$

Recognizability and A_{TM}

Theorem: L is recognizable *if and only if* $L \leq_{\mathrm{m}} A_{\mathrm{TM}}$

Proof:

Example: Another reduction to EQ_{TM}

$$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $A_{\text{TM}} \leq_{\text{m}} EQ_{\text{TM}}$

Proof: The following TM N computes the reduction f:

What should the inputs and outputs to f be?

a) f should take as input a pair $\langle M_1, M_2 \rangle$ and output a pair $\langle M, w \rangle$
b) f should take as input a pair $\langle M, w \rangle$ and output a pair $\langle M_1, M_2 \rangle$
c) f should take as input a pair $\langle M_1, M_2 \rangle$ and either accept or reject
d) f should take as input a pair $\langle M, w \rangle$ and either accept or reject

Example: Another reduction to EQ_{TM}

$EQ_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Theorem: $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

Proof: The following TM computes the reduction f:

On input $\langle M, w \rangle$:

1. Construct TMs M_1, M_2 as follows:

M_1 = "On input x,

M_2 = "On input x,

2. Output $\langle M_1, M_2 \rangle$

Consequences of $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

1. Since A_{TM} is undecidable, EQ_{TM} is also undecidable

2. $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ implies $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} \overline{EQ_{\mathrm{TM}}}$
   Since $\overline{A_{\mathrm{TM}}}$ is unrecognizable, $\overline{EQ_{\mathrm{TM}}}$ is unrecognizable

EQ_{TM} itself is also unrecognizable

$$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $\overline{A_{\text{TM}}} \leq_{\text{m}} EQ_{\text{TM}}$ (Hence EQ_{TM} is unrecognizable)

Proof: The following TM computes the reduction:

On input $\langle M, w \rangle$:

1. Construct TMs M_1, M_2 as follows:

   M_1 = "On input x,
   1. Ignore x
   2. Run M on input w
   3. If M accepts, accept.
      Otherwise, reject."

   M_2 = "On input x,
   1. Ignore x and reject"

2. Output $\langle M_1, M_2 \rangle$