BU CS 332 – Theory of Computation

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Lecture 23:

- More NP-completeness

Reading:

Sipser Ch 7.4-7.5

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NP-completeness

"The hardest languages in NP"

Definition: A language B is NP-complete if

1) $B \in \mathrm{NP}$, and
2) B is NP-hard: Every language $A \in \mathrm{NP}$ is poly-time reducible to B, i.e., $A \leq_{\mathrm{p}} B$

Last time:

$TMSAT = \{\langle N, w, 1^t\rangle$ | NTM N accepts input w within t steps$\}$ is NP-complete

<u>Cook-Levin Theorem:</u>
$\{\langle\varphi\rangle |$ Boolean formula φ is satisfiable$\}$ is NP-complete

New NP-complete problems from old

Lemma: If $A \leq_{\mathrm{p}} B$ and $B \leq_{\mathrm{p}} C$, then $A \leq_{\mathrm{p}} C$

(poly-time reducibility is transitive)

Theorem: If $B \leq_{\mathrm{p}} C$ for some NP-hard language B, then C is also NP-hard

The usual way to prove NP-completeness:

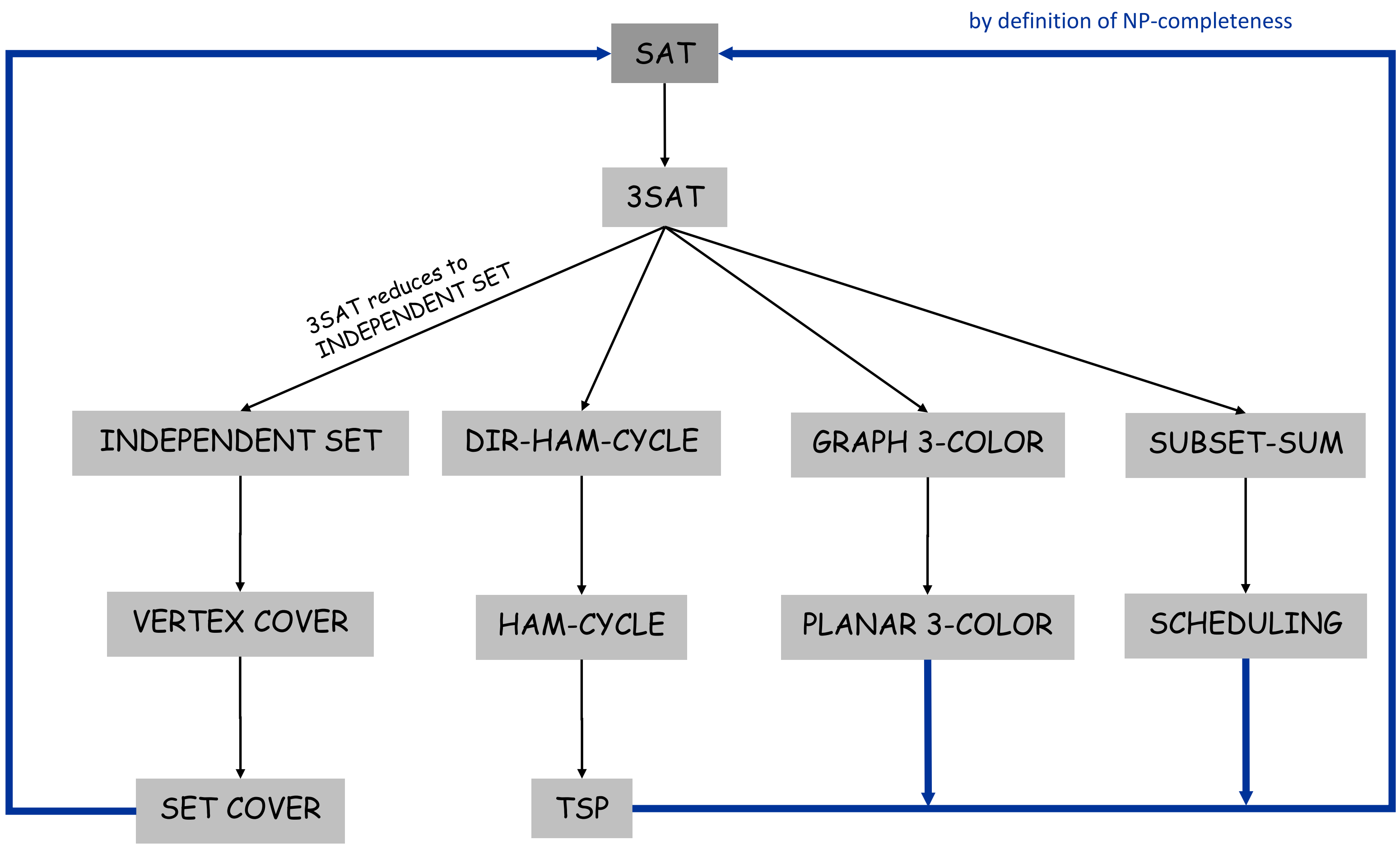
If

1) $C \in \mathrm{NP}$ and
2) There is an NP-complete language B (e.g., 3SAT, VERTEX-COVER, IND-SET, …) such that $B \leq_{\mathrm{p}} C$,

then C is also NP-complete.

New NP-complete problems from old

All problems below are NP-complete and hence poly-time reduce to one another!



$3SAT$ (3-CNF Satisfiability)

Definitions:

- A literal either a variable of its negation x_5 , $\overline{x_7}$
- A clause is a disjunction (OR) of literals Ex. $x_5 \vee \overline{x_7} \vee x_2$
- A 3-CNF is a conjunction (AND) of clauses where each clause contains exactly 3 literals

Ex. $C_1 \wedge C_2 \wedge \ldots \wedge C_m =$

$$(x_5 \vee \overline{x_7} \vee x_2) \wedge (\overline{x_3} \vee x_4 \vee x_1) \wedge \cdots \wedge (x_1 \vee x_1 \vee x_1)$$

$$3SAT = \{\langle\varphi\rangle | 3-\text{CNF formula } \varphi \text{ is satisfiable }\}$$

Some general reduction strategies

- Reduction from special case to general case

  Ex. $VERTEX-COVER \leq_{\mathrm{p}} SET-COVER$

  $3SAT \leq_{\mathrm{p}} SAT$

- Reduction by simple equivalence

  Ex. $IND-SET \leq_{\mathrm{p}} VERTEX-COVER$

  $VERTEX-COVER \leq_{\mathrm{p}} IND-SET$

- "Gadget" reductions

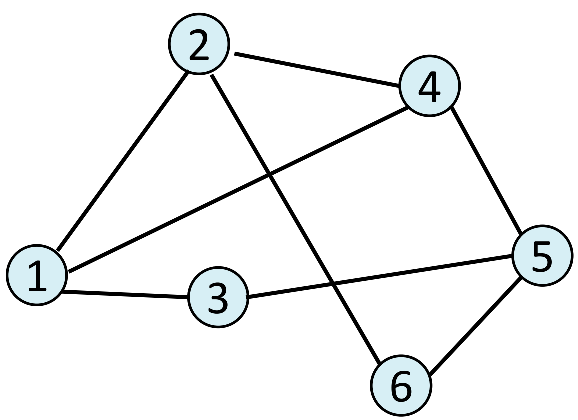
  Ex. $SAT \leq_{\mathrm{p}} 3SAT$

  $3SAT \leq_{\mathrm{p}} IND-SET$

Independent Set

An **independent set** in an undirected graph G is a set of vertices that includes at most one endpoint of every edge.

$$IND-SET = \{\langle G, k\rangle | G \text{ is an undirected graph containing an independent set with} \geq k \text{ vertices}\}$$



Which of the following are independent sets in this graph?

a) $\{1\}$
b) $\{1, 5\}$
c) $\{2, 3, 6\}$
d) $\{3, 4, 6\}$

Independent Set is NP-complete

1) $IND-SET \in \text{NP}$
2) Reduce $3SAT \leq_{\text{p}} IND-SET$

Proof of 1) The following gives a poly-time verifier for $IND-SET$

Certificate: Vertices $v_1, \ldots, v_k$

Verifier:

"On input $\langle G, k; v_1, \ldots, v_k \rangle$, where G is a graph, k is a natural number,

1. Check that $v_1, \ldots v_k$ are distinct vertices in G
2. Check that there are no edges between the v_i's."

Independent Set is NP-complete

1) $IND-SET \in \text{NP}$
2) Reduce $3SAT \leq_{\text{p}} IND-SET$

Proof of 2) The following TM computes a poly-time reduction.

"On input $\langle\varphi\rangle$, where φ is a 3CNF formula,

1. Construct graph G from φ
   - G contains 3 vertices for each clause, one for each literal.
   - Connect 3 literals in a clause in a triangle.
   - Connect every literal to each of its negations.
2. Output $\langle G, k\rangle$, where k is the number of clauses in φ."

Example of the reduction

$$\varphi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_3)$$

Proof of correctness for reduction

Let k = # clauses and l = # literals in φ

Correctness: φ is satisfiable iff G has an independent set of size k

$\Longrightarrow$ Given a satisfying assignment, select one true literal from each triangle. This is an independent set of size k

$\Longleftarrow$ Let S be an independent set in G of size k

- S must contain exactly one vertex in each triangle
- Set these literals to true, and set all other variables arbitrarily
- Truth assignment is consistent and all clauses are satisfied

Runtime: $O(k + l^2)$ which is polynomial in input size

Some general reduction strategies

- Reduction by simple equivalence

  Ex. $IND-SET \leq_{\mathrm{p}} VERTEX-COVER$

  $VERTEX-COVER \leq_{\mathrm{p}} IND-SET$

- Reduction from special case to general case

  Ex. $VERTEX-COVER \leq_{\mathrm{p}} SET-COVER$

  $3SAT \leq_{\mathrm{p}} SAT$

- "Gadget" reductions

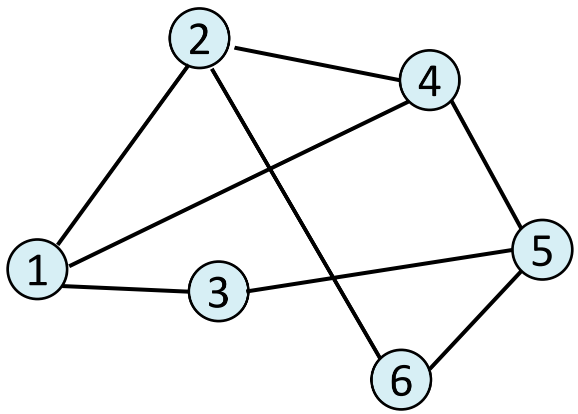
  Ex. $SAT \leq_{\mathrm{p}} 3SAT$

  $3SAT \leq_{\mathrm{p}} IND-SET$

Vertex Cover

Given an undirected graph G, a **vertex cover** in G is a subset of nodes which includes at ***least*** one endpoint of every edge.

$VERTEX - COVER$ = $\{\langle G, k \rangle \mid G$ is an undirected graph which has a vertex cover with $\leq k$ vertices$\}$

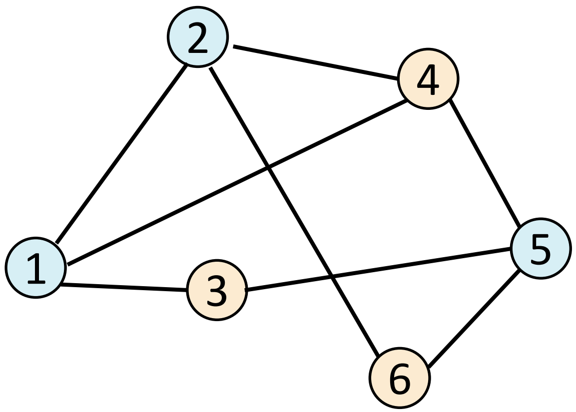


Independent Set and Vertex Cover

Claim. S is an independent set iff $V \setminus S$ is a vertex cover.

$\Longrightarrow$ Let S be any independent set.

- Consider an arbitrary edge (u, v).
- S is independent $\Longrightarrow u \notin S$ or $v \notin S \Longrightarrow u \in V \setminus S$ or $v \in V \setminus S$.
- Thus, $V \setminus S$ covers (u, v).



$\Longleftarrow$ Let $V \setminus S$ be any vertex cover.

- Consider two nodes $u \in S$ and $v \in S$.
- Then $(u, v) \notin E$ since $V \setminus S$ is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Longrightarrow S$ is an independent set.

INDEPENDENT SET reduces to VERTEX COVER

Theorem. IND-SET $\leq_{\mathrm{p}}$ VERTEX-COVER.

What do we need to do to prove this theorem?

a) Construct a poly-time nondet. TM deciding IND-SET
b) Construct a poly-time deterministic TM deciding IND-SET
c) Construct a poly-time nondet. TM mapping instances of IND-SET to instances of VERTEX-COVER
d) Construct a poly-time deterministic TM mapping instances of IND-SET to instances of VERTEX-COVER
e) Construct a poly-time nondet. TM mapping instances of VERTEX-COVER to instances of IND-SET
f) Construct a poly-time deterministic TM mapping instances of VERTEX-COVER to instances of IND-SET

INDEPENDENT SET reduces to VERTEX COVER

Theorem. IND-SET $\leq_{\mathrm{p}}$ VERTEX-COVER.

Proof. The following TM computes the reduction.

"On input $\langle G, k \rangle$, where G is an undirected graph and k is an integer,

1. Output $\langle G, n - k \rangle$, where n is the number of nodes in G."

Correctness:

- G has an independent set of size at least k iff it has a vertex cover of size at most $n - k$.

Runtime:

- Reduction runs in linear time.

VERTEX COVER reduces to INDEPENDENT SET

Theorem. VERTEX-COVER $\leq_{\mathrm{p}}$ IND-SET

Proof. The following TM computes the reduction.

"On input $\langle G, k\rangle$, where G is an undirected graph and k is an integer,

1. Output $\langle G, n-k\rangle$, where n is the number of nodes in G."

Correctness:

- G has a vertex cover of size at most k iff it has an independent set of size at least $n-k$.

Runtime:

- Reduction runs in linear time.

Subset Sum

SUBSET-SUM = $\{\langle w_1, \dots, w_m, t\rangle \mid$ there exists a subset of natural numbers $w_1, \dots, w_m$ that sum to $t\}$

Theorem: SUBSET-SUM is NP-complete

Claim 1: SUBSET-SUM is in NP

Claim 2: SUBSET-SUM is NP-hard

$3\text{SAT} \leq_p$ SUBSET–SUM

Goal: Given a 3CNF formula φ on v variables and k clauses, construct a SUBSET-SUM instance $w_1, \ldots, w_m, t$ such that

φ is satisfiable iff there exists a subset of $w_1, \ldots, w_m$ that sum to t

First attempt: Encode each literal ℓ of φ as a k-digit *decimal* number $w_\ell = c_1 \ldots c_k$ where

$$c_i = \begin{cases} 1 \text{ if } \ell \text{ appears in clause } i \\ 0 \text{ otherwise} \end{cases}$$

Example of the first attempt reduction

$\varphi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3)$

$3\text{SAT} \leq_p$ SUBSET−SUM

First attempt: Encode each literal ℓ of φ as a k-digit *decimal* number $w_\ell = c_1 \dots c_k$ where

$$c_i = \begin{cases} 1 \text{ if } \ell \text{ appears in clause } i \\ \quad 0 \text{ otherwise} \end{cases}$$

Claim: If φ is satisfiable, then there exists a subset of the w_ℓ's that "digit-wise" add up to "at least" $111 \dots 11$

Two issues:

1) Need to enforce that exactly one of $\ell, \overline{\ell}$ is set to 1
2) Need the subset to add up to exactly some target

$3\text{SAT} \leq_p$ SUBSET−SUM

Actual reduction: Encode each literal ℓ of φ as a $(v + k)$-digit *decimal* number $w_\ell = b_1 \ldots b_v |\ c_1 \ldots c_k$ where

$$b_i = \begin{cases} 1 \text{ if } \ell \in \{x_i, \overline{x_i}\} \\ 0 \text{ otherwise} \end{cases} \qquad c_i = \begin{cases} 1 \text{ if } \ell \text{ appears in clause } i \\ 0 \text{ otherwise} \end{cases}$$

Also, include two copies each of 000…0|100…0, 000…0|010…0, … 000…0|0…01

Claim: φ is satisfiable if and only if there exists a subset of the numbers that add up to $t = 111 \ldots 11|333 \ldots 33$

Example of the reduction

$$\varphi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3)$$

Encode literal ℓ as $w_\ell = b_1 \dots b_v | \, c_1 \dots c_k$ where

$$b_i = \begin{cases} 1 \text{ if } \ell \in \{x_i, \overline{x_i}\} \\ 0 \text{ otherwise} \end{cases}$$

$$c_i = \begin{cases} 1 \text{ if } \ell \text{ appears in clause } i \\ 0 \text{ otherwise} \end{cases}$$

Include two copies each of 000…0|100…0, 000…0|010…0, … 000…0|0…01