Homework 8 – Due Tuesday, April 1, 2025 at 11:59 PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators and write "Collaborators: none" if you worked by yourself. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Note You may use various generalizations of the Turing machine model we have seen in class, such as TMs with two-way infinite tapes, stay-put, or multiple tapes. If you choose to use such a generalization, state clearly and precisely what model you are using. **You may describe Turing machines at a high-level on this assignment.**

Problems There are 5 required problems and one bonus problem.

1. (**Reduction Mad-Libs**) A language B is *excited* if every string in B takes the form ww for some $w \in \{0,1\}^*$. For example, $\{00, 1111, 1010\}$ is excited, but $\{00, 10\}$ is not excited. The language $EX_{\text{TM}} = \{\langle M \rangle \mid L(M) \text{ is excited}\}$ corresponds to the following computational problem: Given the encoding of a TM M, does M recognize an excited language? This exercise will walk you through a proof, by reduction, that EX_{TM} is undecidable.

   Assume, for the sake of contradiction, that EX_{TM} is decidable by a TM R. That is, there is a TM R that accepts $\langle M \rangle$ whenever $L(M)$ is excited, and rejects $\langle M \rangle$ whenever $L(M)$ is not excited. We will use R to construct a new TM S that decides the (undecidable) language A_{TM}.

   > **Algorithm:** $S(\langle M, w \rangle)$
   >
   > **Input** : Encoding of a basic TM M over input alphabet $\{0,1\}$, string $w \in \{0,1\}^*$
   >
   > 1. Construct the following TM N:
   > $N =$ "On input a string $x \in \{0,1\}^*$:
   >   If $x = 00$, *accept*.
   >   Else, run M on input w. If it accepts, *accept*. Otherwise, *reject*."
   > 2. Run R on input $\langle N \rangle$. If it accepts, __(i)__. Otherwise, __(ii)__.

   (a) Consider the machine N constructed inside algorithm S. If M accepts on input w, what is the language $L(N)$? Is $L(N)$ excited in this case?

   (b) If M does not accept on input w, what is the language $L(N)$? Is $L(N)$ excited in this case?

   (c) Fill in the blanks labeled (i) and (ii) with *accept* or *reject* decisions to guarantee the following conditions: If M accepts input w, then S accepts input $\langle M, w \rangle$, and if M does not accept input w, then S rejects input $\langle M, w \rangle$. Use parts (a) and (b) to explain why these conditions hold for your choices of how to fill in the blanks.

   (Your job is done now, but you may want to keep reading to see the exciting conclusion.) By part (c), the TM M exactly decides the language A_{TM}. But this language is undecidable, which is a contradiction. Hence our assumption that EX_{TM} was decidable is false, so we conclude that EX_{TM} is an undecidable language.

2. (**Unary Acceptance, Redux**) A Turing machine M *accepts a unary string* if there exists a string $x \in \{1\}^*$ such that M accepts on input x. Consider the problem of determining whether (the encoding of) a TM M with input alphabet $\{0, 1\}$ accepts a unary string. On Homework 7, you showed that the corresponding language $UA = \{\langle M \rangle \mid \text{ TM } M \text{ accepts on input } 1^n \text{ for some } n \geq 0\}$ is Turing-recognizable.

   Prove that the language UA is undecidable.

   Hint: Give a reduction from the undecidable language A_{TM}. That is, you should assume for the sake of contradiction that UA is decidable. Then under this assumption, construct a TM deciding A_{TM}, prove that this decider is correct, and as a result conclude that your assumption that UA is decidable must have been false.

3. (**Odd-Length TM**) Let $OL_{\text{TM}} = \{\langle M \rangle | \ M \text{ is a TM that accepts all strings whose length is an odd number and rejects all other strings}\}$.

   (a) Your first goal in this problem is to use a mapping reduction from $\overline{A_{\text{TM}}}$ to show that OL_{TM} is not Turing-recognizable. Describe what the inputs and outputs of this mapping reduction should be.

   (b) Describe a TM computing a mapping reduction from $\overline{A_{\text{TM}}}$ to OL_{TM} and explain why this TM is correct.

   (c) Explain why part (b) implies that OL_{TM} is not Turing-recognizable.

   (d) Use a (different) mapping reduction to prove that $\overline{OL_{\text{TM}}}$ is not Turing-recognizable (i.e., OL_{TM} is not co-Turing-recognizable). Explain briefly why the TM computing your mapping reduction is correct.

4. (**Complementary TMs**) Consider the following computational problem. Given two TMs M_1 and M_2, is it the case that the language recognized by M_1 is exactly the complement of the language recognized by M_2?

   (a) Formulate this problem as a language C_{TM}.

   (b) Prove that C_{TM} is undecidable. You may either use an "informal" reduction as in Chapter 5.1 of Sipser, or a mapping reduction as in Chapter 5.3 – it's your choice. In either case, explain (briefly) why the TM computing your reduction is correct.

5. (**Asymptotic Notation Review**) This problem will be graded automatically by Gradescope. Please enter your answers manually by completing the assignment Homework 8-Problem 5. For each of the following, select *true* or *false* using the radio buttons on Gradescope. All logarithms are base 2 unless otherwise stated.

   (a) $1337 = O(n^2)$

   (b) $n^{10} = O(n^9 \log n)$

   (c) $n \log n + 10n = o(n^2)$

   (d) $4^n = O(2^n)$

   (e) $4^n = 2^{O(n)}$

   (f) $2^{2^n} = O(2^{n^2})$

   (g) $2n = o(n^2)$

   (h) $10^6 = o(n)$

   (i) $2 \log n = o(\log n)$

   (j) $\frac{1}{3} = o(1)$

   (k) $2^n = o(4^n)$

   (l) $n^5 = O(32^{\log_2 n})$

   (m) $\log n = O(\log(\log n))$

   (n) $3^{2^n} = o(2^{3^n})$

6. (**Bonus problem**) Define the language $XOR_{\mathrm{TM}} = \{\langle M, w, v\rangle \mid M$ is a TM that accepts exactly one of the strings $w, v\}$. Prove that both XOR_{TM} and its complement $\overline{XOR_{\mathrm{TM}}}$ are unrecognizable.