BU CS 332 – Theory of Computation

Link to polls:
https://forms.gle/eeRKEZf5phJ3GBZr5

Lecture 2:

- Parts of a Theory of Computation
- Sets, Strings, and Languages

Reading:

Sipser Ch 0

Reminders:

- HW1 due tomorrow night (Tue, 11:59PM)

Mark Bun

January 27, 2025

What makes a good theory?

- General ideas that apply to many different systems
- Expressed simply, abstractly, and precisely

Parts of a Theory of Computation

- Models for machines (computational devices)
- Models for the problems machines can be used to solve
- Theorems about what kinds of machines can solve what kinds of problems, and at what cost

What is a (Computational) Problem?

For us: A problem will be the task of determining whether a *string* is in a *language*

E.g. Parity: Given a string of a's and b's, does it contain an even number of a's?

What is a (Computational) Problem?

For us: A problem will be the task of determining whether a *string* is in a *language*

- **Alphabet**: A finite set Σ Ex. $\Sigma = \{\mathrm{a}, \mathrm{b}\}$
- **String:** A finite concatenation of alphabet symbols

  Ex. $\mathrm{bba}, \mathrm{ababb}$

  ε denotes empty string, length 0

  Σ^* = set of all strings using symbols from Σ

  Ex. $\{\mathrm{a}, \mathrm{b}\}^* = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{bb}, \ldots\}$
- **Language:** A set $L \subseteq \Sigma^*$ of strings

Examples of Languages

Parity: Given a string consisting of a's and b's, does it contain an even number of a's?

$\Sigma = \{\mathrm{a}, \mathrm{b}\}$ $\quad L = \{x \in \{\mathrm{a}, \mathrm{b}\}^* \mid x$ has an even # of a's$\}$

Primality: Given a natural number x (represented in binary), is x prime?

$\Sigma = \{0, 1\}$ $\quad L = \{x \in \{0, 1\}^* \mid x$ is the binary rep. of a prime$\}$

Halting Problem: Given a C program, can it ever get stuck in an infinite loop?

$\Sigma =$ Extended ASCII $\quad L = \{P \in \Sigma^* \mid P$ describes a C program that loops forever on some input$\}$

Primality language

Which best represents the language corresponding to the Primality problem? (I.e., strings over the alphabet $\{0, 1\}$ that are binary representations of prime numbers.)

Let's say the most significant bit is on the left, so "100" is the binary representation of 4.

(a) $\{2, 3, 5, 7, \ldots\}$

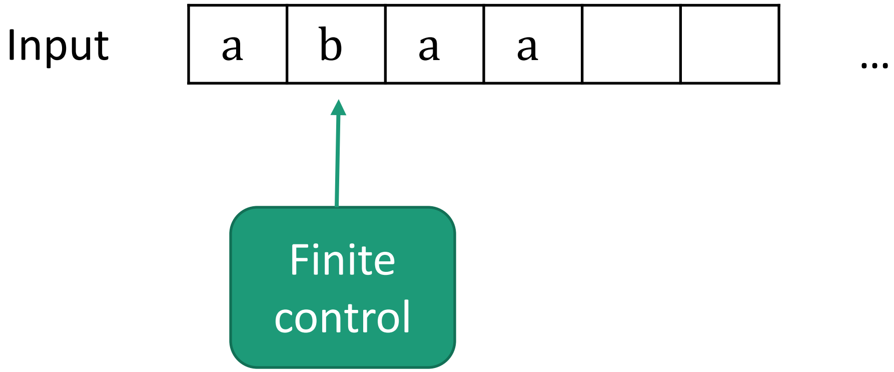
(b) $\{10, 11, 101, 111, \ldots\}$

(c) $\{11, 111, 11111, 1111111, \ldots\}$

(d) $\{11, 011, 101, 110, 111, 0111, \ldots\}$

Machine Models

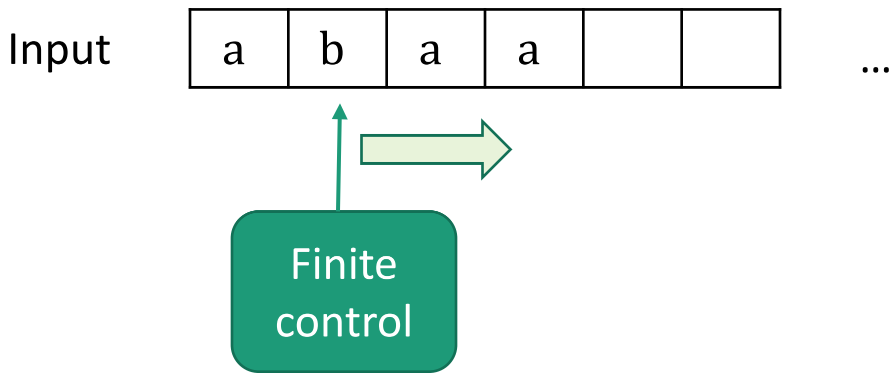
Computation is the processing of information by the **unlimited application** of a **finite set** of operations or rules



Abstraction: We don’t care how the control is implemented. We just require it to have a finite number of states, and to transition between states using fixed rules.

Machine Models

- <u>Finite Automata (FAs):</u> Machine with a finite amount of unstructured memory

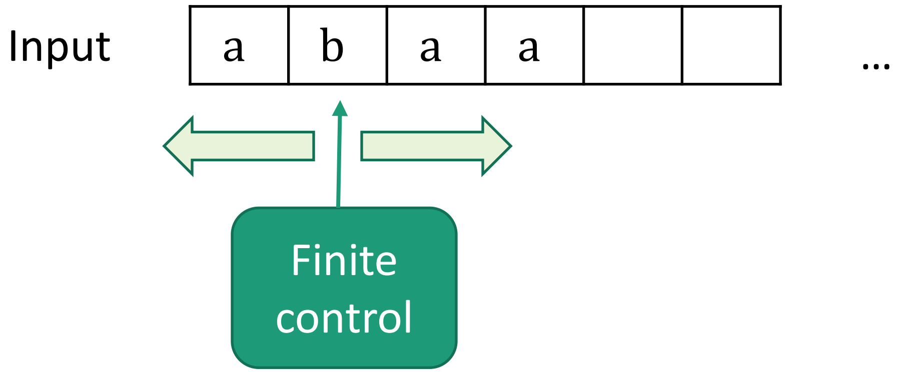


Control scans left-to-right
Can check simple patterns
Can't perform unlimited counting

Useful for modeling chips, simple control systems, choose-your-own adventure games…

Machine Models

- <u>Turing Machines (TMs):</u> Machine with unbounded, unstructured memory



Control can scan in both directions
Control can both read and <u>write</u>

Model for general sequential computation
Church-Turing Thesis: Everything we intuitively think of as "computable" is computable by a Turing Machine

What theorems would we like to prove?

We will define classes of languages based on which machines can solve the associated computational problems

Inclusion: Every language recognizable by a FA is also recognizable by a TM

Non-inclusion: There exist languages recognizable by TMs which are not recognizable by FAs

Completeness: Identify a “hardest” language in a class

Robustness: Alternative definitions of the same class

Ex. Languages recognizable by FAs = regular expressions

Why study theory of computation?

- You’ll learn how to formally reason about computation
- You’ll learn the technology-independent foundations of CS

Philosophically interesting questions:

- Are there well-defined problems which cannot be solved by computers?
- Can we always find the solution to a puzzle faster than trying all possibilities?
- Can we say what it means for one problem to be “harder” or “no harder” than another?

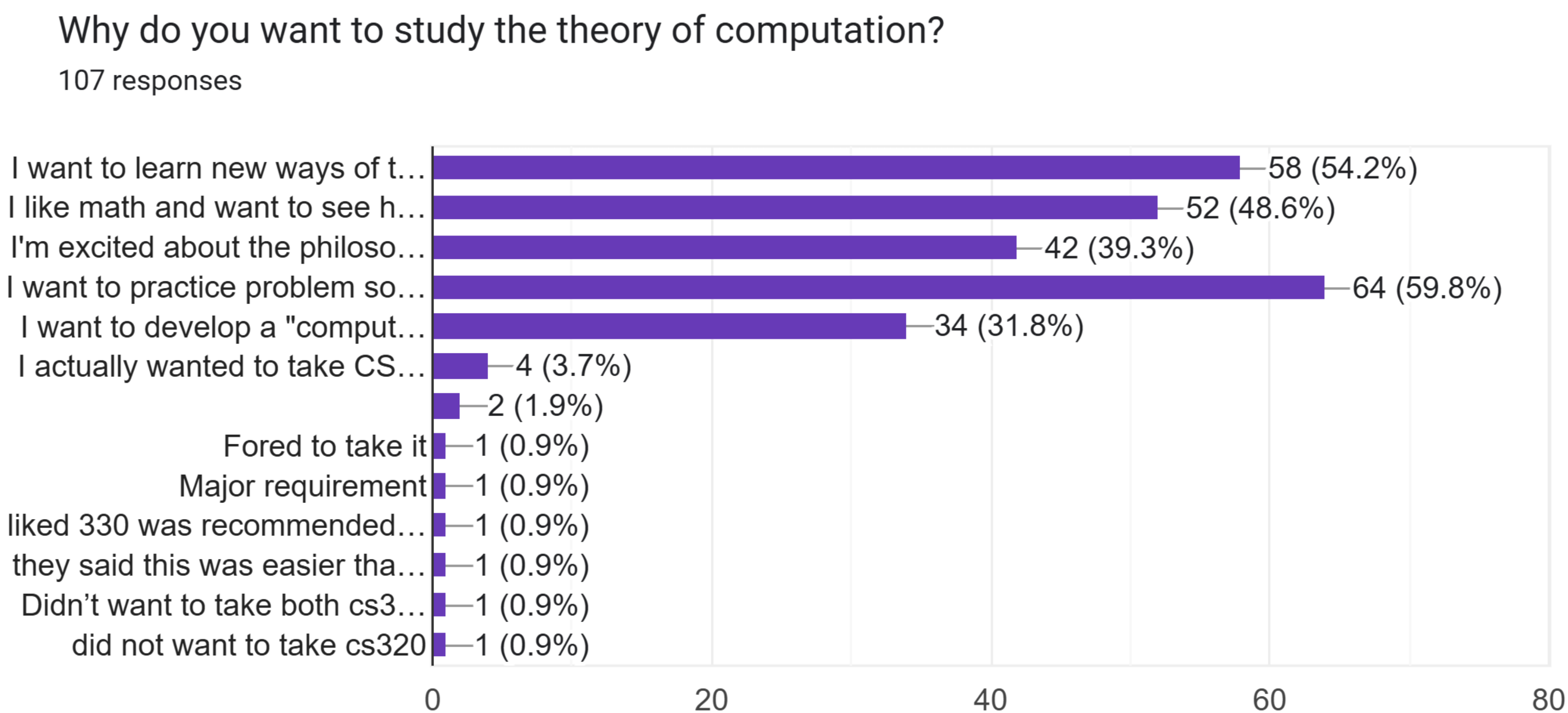
Why study theory of computation?

- You’ll learn how to formally reason about computation
- You’ll learn the technology-independent foundations of CS

Connections to other parts of science:

- Finite automata arise in compilers, AI, coding, chemistry
  https://cstheory.stackexchange.com/a/14818
- Hard problems are essential to cryptography
- Computation occurs in cells/DNA, the brain, economic systems, physical systems, social networks, etc.

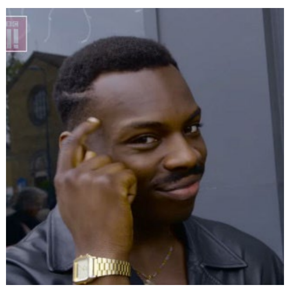
What appeals to you about the theory of computation?



Why study theory of computation?

Practical knowledge for developers


"Boss, I can't find an efficient algorithm. I guess I'm just too dumb."

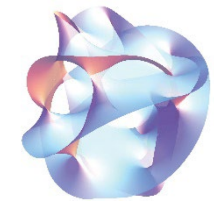
"Boss, I can't find an efficient algorithm because no such algorithm exists."

Will you be asked about this material on job interviews?

No promises, but a true story...

More about strings and languages

String Theory



- **Symbol:** Ex. $\mathrm{a, b, 0, 1}$
- **Alphabet**: A finite set Σ of symbols Ex. $\Sigma = \{\mathrm{a, b}\}$
- **String:** A finite concatenation of alphabet symbols

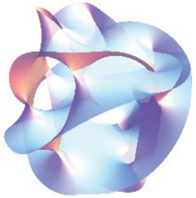
  Ex. $\mathrm{bba, ababb}$

  ε denotes empty string, length 0

  Σ^* = set of all strings using symbols from Σ

  Ex. $\{\mathrm{a, b}\}^* = \{\varepsilon, \mathrm{a, b, aa, ab, ba, bb}, \dots\}$
- **Language:** A set $L \subseteq \Sigma^*$ of strings

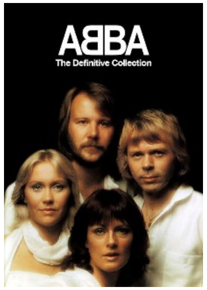
String Theory



- **Length** of a string, written $|x|$, is the number of symbols

Ex. $|\mathrm{abba}| =$ $\qquad$ $|\varepsilon| =$

- **Concatenation** of strings x and y, written xy, is the symbols from x followed by the symbols from y



Ex. $x = \mathrm{ab}, y = \mathrm{ba} \quad \Rightarrow \quad xy =$

$x = \mathrm{ab}, y = \varepsilon \quad \Rightarrow \quad xy =$

- **Reversal** of string x, written x^R, consists of the symbols of x written backwards

Ex. $x = \mathrm{aab} \quad \Rightarrow \quad x^R =$

Fun with String Operations

What is $(xy)^R$?

Ex. $x = \text{aba}, y = \text{bba}$ $\Rightarrow$ $xy =$

$\Rightarrow$ $(xy)^R =$

a) $x^R y^R$

b) $y^R x^R$

c) $(yx)^R$

d) xy^R

Fun Proofs with String Operations

Claim: $(xy)^R =$

Proof: Let $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_m$

Then $(xy)^R =$

Not even the most formal way to do this:

1. Define string length recursively
2. Prove by induction on $|y|$

Languages

A language L is a set of strings over an alphabet Σ

i.e., $L \subseteq \Sigma^*$

Languages = computational (decision) problems

<u>Input:</u> String $x \in \Sigma^*$

<u>Output:</u> Is $x \in L$? (Yes or No?)

Some Simple Languages

$\Sigma = \{0, 1\}$ $\Sigma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

$\emptyset$ (Empty set)

Σ^* (All strings)

$\Sigma^n = \{x \in \Sigma^* \mid |x| = n\}$
(All strings of length n)

Some More Interesting Languages

- $L_1 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's

- $L_2 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that start with (0 or more) a's and are followed by an equal number of b's

- $L_3 =$ The set of strings $x \in \{0, 1\}^*$ that contain the substring "0100"

Some More Interesting Languages

- $L_4 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ of length at most 4

- $L_5 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that contain at least two a's

New Languages from Old

$L_6 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's and length greater than 4

Since languages are just sets of strings, can build them using set operations:

$A \cup B$ "union"

$A \cap B$ "intersection"

$\bar{A}$ "complement"

New Languages from Old

$L_6 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's and have length greater than 4

- $L_1 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ that have an equal number of a's and b's
- $L_4 =$ The set of strings $x \in \{\mathrm{a}, \mathrm{b}\}^*$ of length at most 4

$$\Rightarrow L_6 =$$

Operations Specific to Languages

- **Reverse**: $L^R = \{x^R | \, x \in L\}$

  Ex. $L = \{\varepsilon, \mathrm{a}, \mathrm{ab}, \mathrm{aab}\}$ $\Rightarrow L^R =$

- **Concatenation:** $L_1 \circ L_2 = \{xy \, | x \in L_1, y \in L_2\}$

  Ex. $L_1 = \{\mathrm{ab}, \mathrm{aab}\}$ $L_2 = \{\varepsilon, \mathrm{b}, \mathrm{bb}\}$

  $\Rightarrow L_1 \circ L_2 =$

A Few "Traps"

String, language, or something else?

ε

$\emptyset$

$\{\varepsilon\}$

$\{\emptyset\}$

Languages

Languages = computational (decision) problems

Input: String $x \in \Sigma^*$

Output: Is $x \in L$? (Yes or No? I.e., Accept or Reject?)

The language **recognized** by a program is the set of strings $x \in \Sigma^*$ that it *accepts*

What Language Does This Program Recognize?

Alphabet $\Sigma = \{\mathrm{a}, \mathrm{b}\}$

On input $x = x_1 x_2 \dots x_n$:

count = 0

  For $i = 1, \dots, n$:

    If $x_i = \mathrm{a}$:

      count $=$ count $+ 1$

  If count ≤ 4: accept

  Else: reject

a) $\{x \in \Sigma^* \mid |x| > 4\}$
b) $\{x \in \Sigma^* \mid |x| \leq 4\}$
c) $\{x \in \Sigma^* \mid |x| = 4\}$
d) $\{x \in \Sigma^* \mid x \text{ has more than } 4 \text{ a's}\}$
e) $\{x \in \Sigma^* \mid x \text{ has at most } 4 \text{ a's}\}$
f) $\{x \in \Sigma^* \mid x \text{ has exactly } 4 \text{ a's}\}$