BU CS 332 – Theory of Computation

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Lecture 5:

- Closure Properties
- Regular Expressions

Reading:

Sipser Ch 1.2-1.3

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Last Time

- Nondeterministic Finite Automata
- NFAs vs. DFAs
  - Subset construction: NFA $\rightarrow$ DFA

Closure Properties

An Analogy

In algebra, we try to identify operations which are common to many different mathematical structures

Example: The integers $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ are **closed** under

- Addition: $x + y$
- Multiplication: $x \times y$
- Negation: $-x$
- ...but NOT Division: $x \,/\, y$

$7 + (-4) = 3 \in \mathbb{Z}$

$7 \times (-4) = -28 \in \mathbb{Z}$

$-(-4) = 4 \in \mathbb{Z}$

$2/3 \in \mathbb{Z}$

We'd like to investigate similar closure properties of the class of regular languages

Regular operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

$A = \{11\}$

$A^* = \{\varepsilon, 11, 1111,$
$111111, \ldots\}$

Union: $A \cup B = \{w \mid w \in A \textbf{ or } w \in B\}$

$B = \{00, 1\}$

$B^* = \{\varepsilon, 00, 1,$
$0000, 001,$
$100, 11,$
$000000,$
$00001, \ldots$

Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$

Star: $A^* = \{a_1 a_2 \ldots a_n \mid n \geq 0, \text{ each } a_i \in A\}$

$= \{\varepsilon\} \cup A \cup (A \circ A) \cup (A \circ A \circ A) \cup \ldots$

Check: For any finite set of single characters Σ,
this definition agrees w/ our previous definition of Σ^*

Other operations

Let $A, B \subseteq \Sigma^*$ be languages. Define

Complement: $\bar{A} = \{w \mid w \notin A\}$

Intersection: $A \cap B = \{w \mid w \in A \textbf{ and } w \in B\}$

Reverse: $A^R = \{w \mid w^R \in A\}$

Operations on languages

Let $A, B \subseteq \Sigma^*$ be languages. Define

Regular Operations

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Concatenation: $A \circ B = \{xy \mid x \in A, y \in B\}$
- Star: $A^* = \{ w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in A\}$

Complement: $\bar{A} = \{x \mid x \notin A\}$

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Reverse: $A^R = \{ a_1 a_2 \ldots a_n \mid a_n \ldots a_1 \in A\}$

Theorem: The class of regular languages is closed under all six of these operations, i.e., if A and B are regular, applying any of these operations yields a regular language

Proving Closure Properties

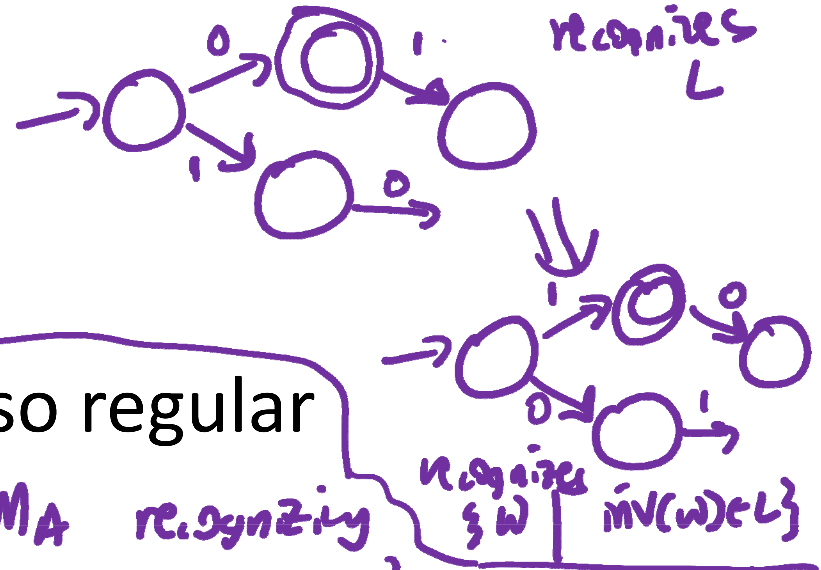
Complement

Complement: $\bar{A} = \{ w \mid w \notin A \}$

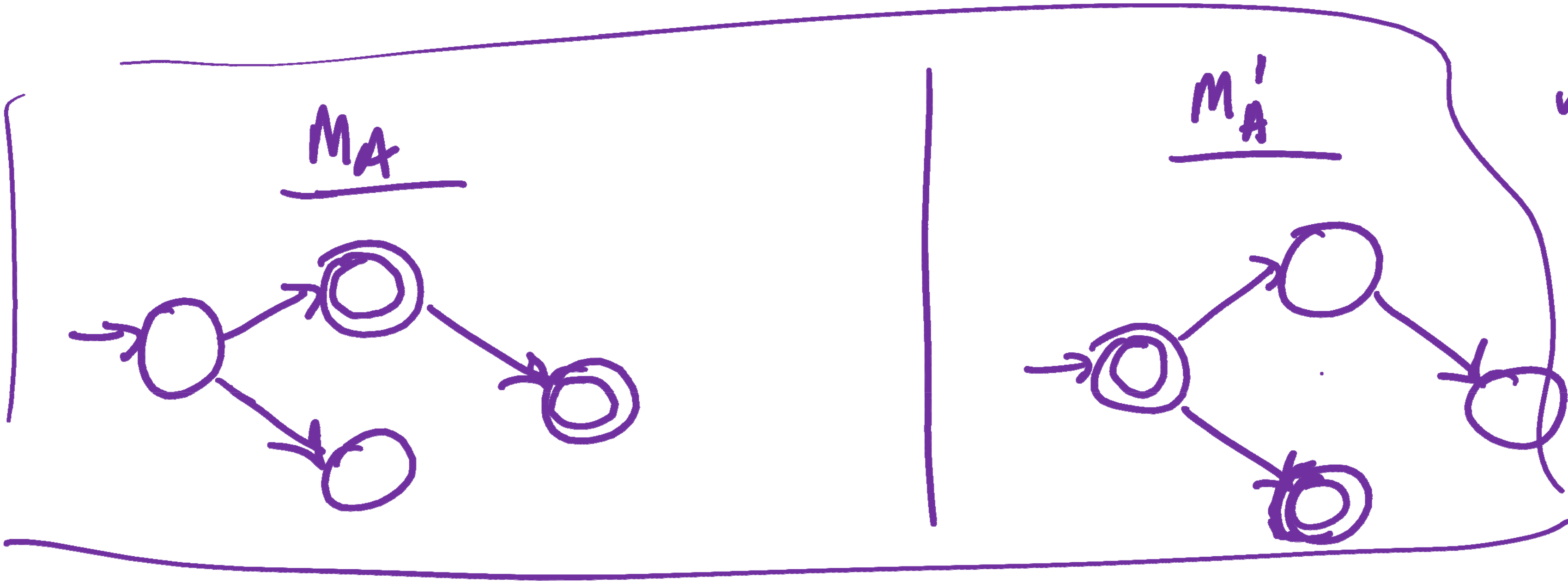
Theorem: If A is regular, then $\bar{A}$ is also regular

Proof idea: A regular $\Rightarrow$ $\exists$ a DFA M_A recognizing A

Goal: Use M_A to construct a DFA M'_A recognizing $\bar{A}$



Construct M'_A by switching accept and reject states of M_A



where inv(w) = the string where all 0's in w are replaced by 1's and vice versa

Complement, Formally

Let $M = (Q, \Sigma, \delta, q_0, \boxed{F})$ be a DFA recognizing a language A. Which of the following represents a DFA recognizing $\bar{A}$?

Switch accept states F of M w/
reject states $Q \setminus F$

a) $(F, \Sigma, \delta, q_0, Q)$

b) $(Q, \Sigma, \delta, q_0, \boxed{Q \setminus F})$, where $Q \setminus F$ is the set of states in Q that are not in F

c) $(Q, \Sigma, \delta', q_0, F)$ where $\delta'(q, s) = p$ such that $\delta(p, s) = q$

d) None of the above

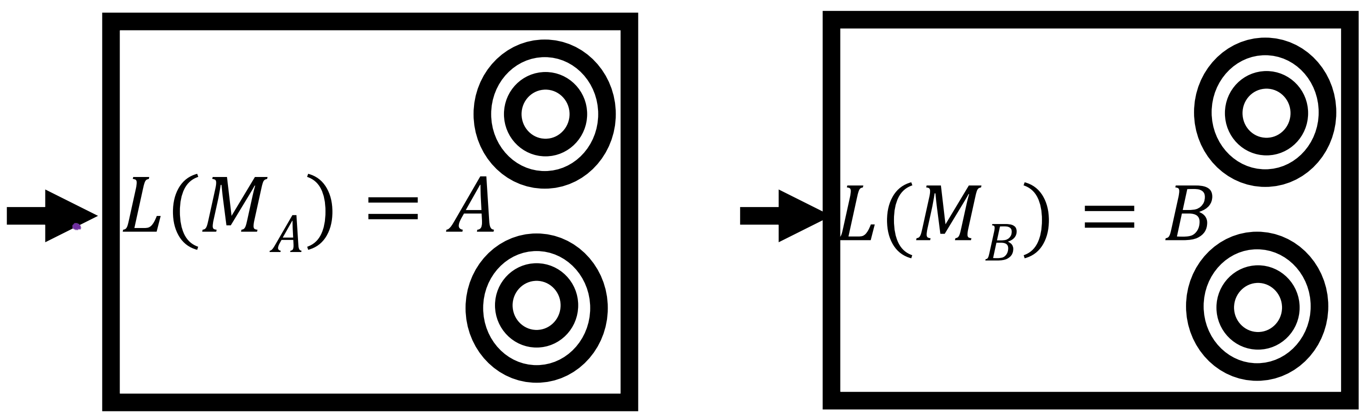
Closure under Concatenation

Concatenation: $A \circ B \;=\; \{\, xy \mid x \in A, y \in B \,\}$

Theorem. If A and B are regular, then $A \circ B$ is also regular.

Proof idea: Given DFAs M_A and M_B, construct an NFA for $A \circ B$ by

- Connecting all accept states in M_A to the start state in M_B.
- Making all states in M_A non-accepting.



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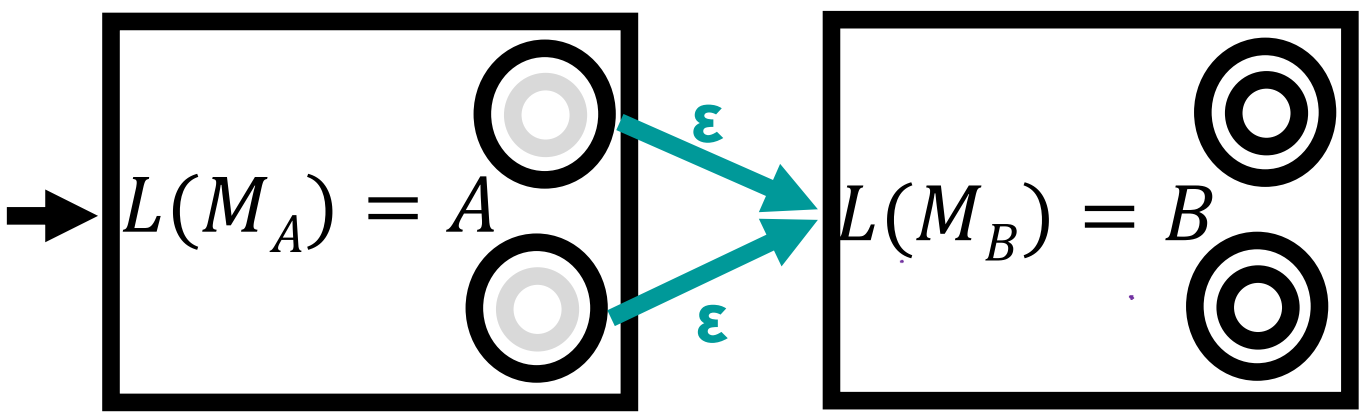
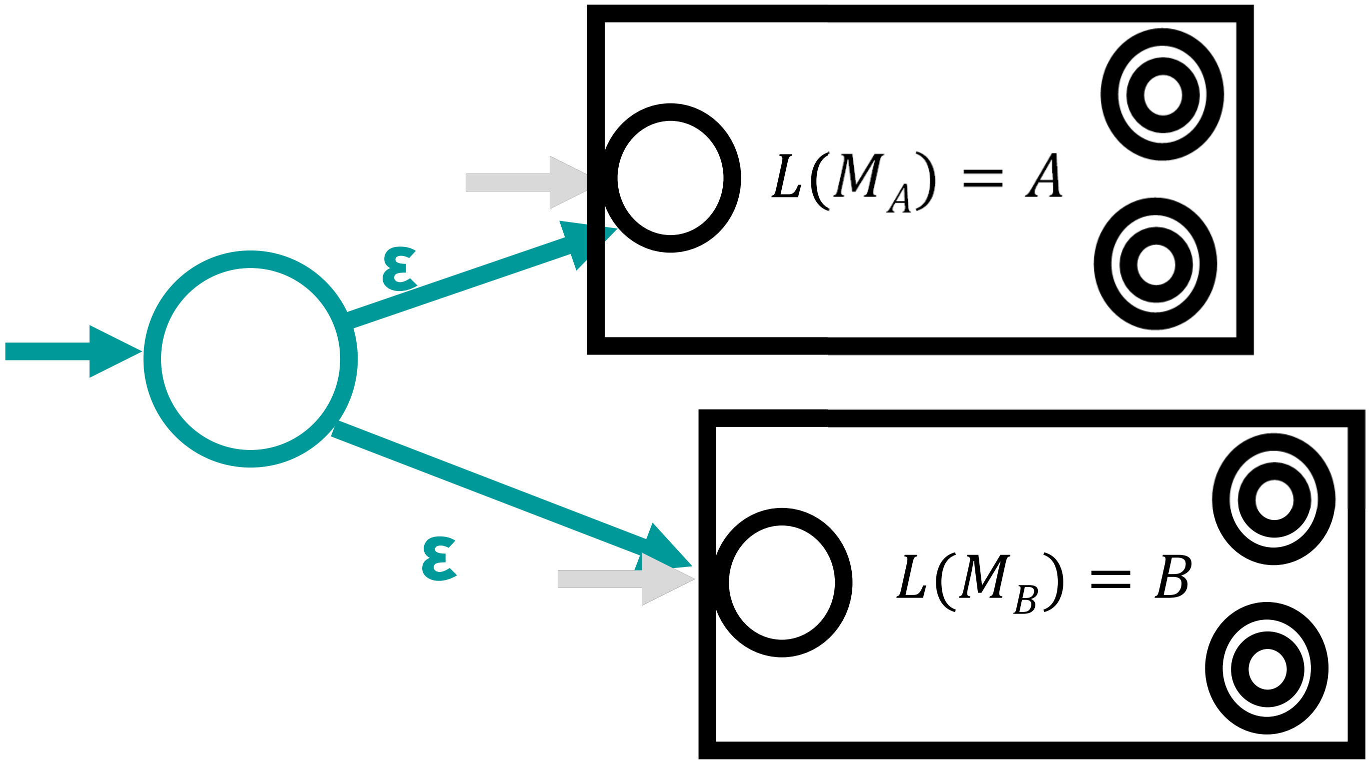


A Mystery Construction

Given DFAs M_A recognizing A and M_B recognizing B, what does the following NFA recognize?

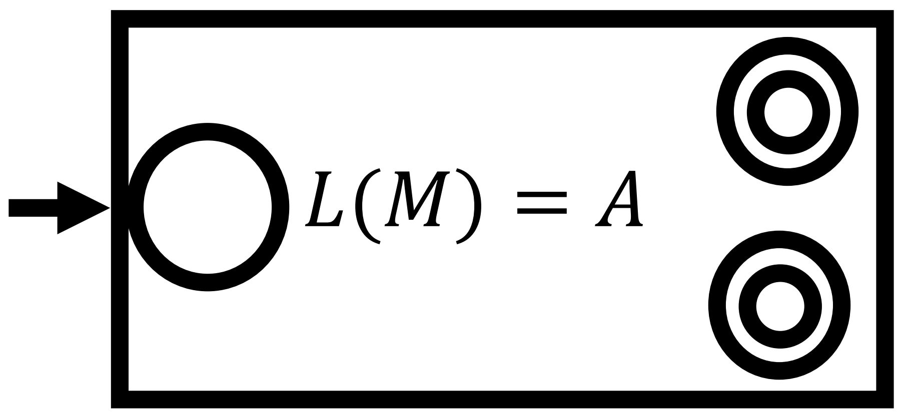


a) $A \cup B$
b) $A \circ B$
c) $A \cap B$
d) $\{\varepsilon\} \cup A \cup B$

Closure under Star

Star: $A^* = \{\, a_1 a_2 \ldots a_n \,|\, n \geq 0 \text{ and each } a_i \in A\}$

Theorem. If A is regular, then A^* is also regular.

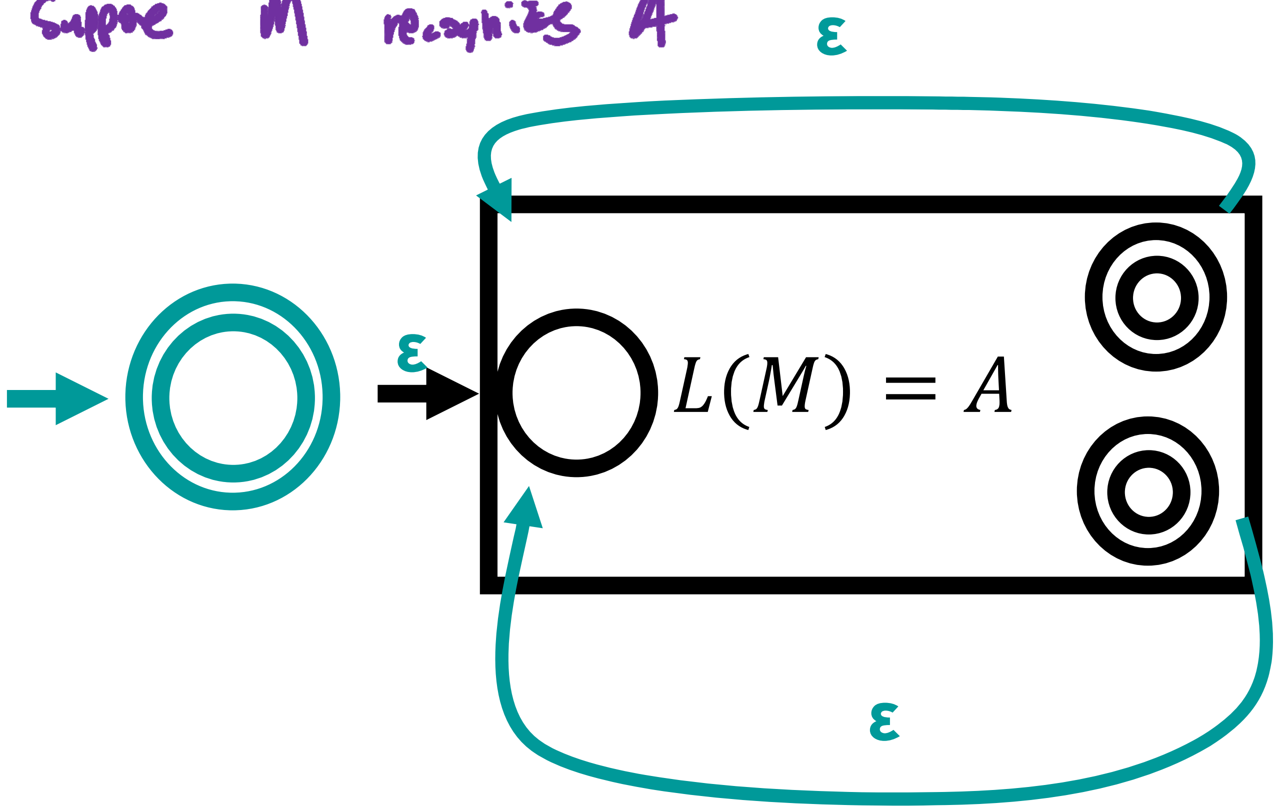


Closure under Star

Star: $A^* = \{ a_1 a_2 ... a_n \,|\, n \geq 0 \text{ and each } a_i \in A\}$

Theorem. If A is regular, then A^* is also regular.

Suppose M recognizes A



On proving your own closure properties

You'll have homework/test problems of the form "show that the regular languages are closed under some operation"

op (A, B) takes languages A, B, produces some new language

To show regular langs are closed under op, WTS $\forall$ regular A, B op(A, B) is also regular

What would Sipser do?

- Give the "proof idea": Explain how to take machine(s) recognizing regular language(s) and create a new machine
- Explain in a few sentences why the construction works
- Give a formal description of the construction
- No need to formally prove that the construction works

Regular Expressions

Regular Expressions

- A different way of describing regular languages
- A regular expression expresses a (possibly complex) regular language by combining simple languages using the regular operations

"Simple" languages: $\emptyset, \{\varepsilon\}, \{a\}$ for some $a \in \Sigma$

Regular operations:

Union: $A \cup B$

Concatenation: $A \circ B = \{ab \mid a \in A, b \in B\}$

Star: $A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0 \text{ and } a_i \in A\}$

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are
   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$)

$(a \circ b)$ $\quad$ $((((a \circ (b^*)) \circ c) \cup (((a^*) \circ b))^*))$ $\quad$ $(\emptyset^*)$

$L(M)$ = language recognized by DFA M

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\big((R_1^*)\big) = (L(R_1))^*$

Regular Expressions – Example

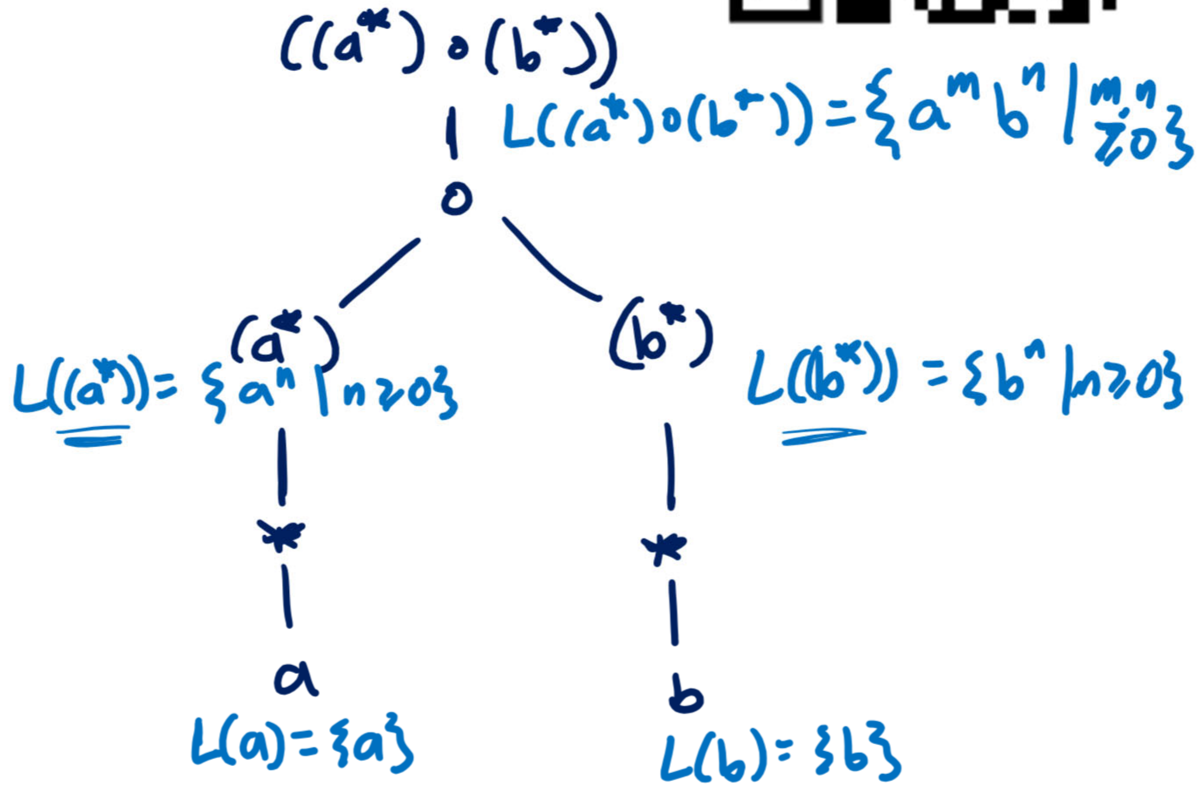
$L(((a^*) \circ (b^*))) =$

a) $\{a^n b^n \mid n \geq 0\}$

b) $\{a^m b^n \mid m, n \geq 0\}$

c) $\{(ab)^n \mid n \geq 0\}$

d) $\{a, b\}^*$



$A^* = \{ a_1 a_2 \ldots a_n \mid n \geq 0,\ a_i \in A \text{ for each } i \}$

Simplifying Notation

- Omit $\circ$ symbol: $(ab) = (a \circ b)$
- Omit many parentheses, since union and concatenation are associative:

$$(a \cup b \cup c) = (a \cup (b \cup c)) = ((a \cup b) \cup c)$$

- Order of operations: Evaluate star, then concatenation, then union

$$ab^* \cup c = (a(b^*)) \cup c$$

Examples

$L((0 \cup 1)^*)$
$= (L(0 \cup 1))^* = (\{0,1\})^*$

Let $\Sigma = \{0, 1\}$

1. $\{w \mid w$ contains exactly one $1\}$ $\boxed{0^* 1 0^*}$

   $\{x1y \mid x, y \in \{0\}^*\}$

2. $\{w \mid w$ has length at least 3 and its third symbol is $0\}$

   $\Sigma \Sigma 0 \Sigma^*$ i.e. $(0 \cup 1)(0 \cup 1) 0 (0 \cup 1)^*$

3. $\{w \mid$ every odd position of w is $1\}$

   $(1(0 \cup 1))^* 1^*$ also $(1(0 \cup 1))^* (\varepsilon \cup 1)$

   ε 10 11 1010, 1011, 1110, 1111,

Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$

- For regex R, the regex $R^{+} = RR^{*}$