BU CS 332 – Theory of Computation

https://forms.gle/xcJwrTwrLQuitQrj9

Lecture 6:

- Regexes = NFAs
- Limitations of Finite Automata

Reading:

Sipser Ch 1.3

“Myhill-Nerode” note

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February 10, 2025

Regular Expressions – Syntax

A regular expression R is defined recursively using the following rules:

1. ε, $\emptyset$, and a are regular expressions for every $a \in \Sigma$

2. If R_1 and R_2 are regular expressions, then so are

   $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*)

Examples: (over $\Sigma = \{a, b, c\}$) (with simplified notation)

ab $ab^*c \cup (a^*b)^*$ $\emptyset$

Regular Expressions – Semantics

$L(R)$ = the language a regular expression describes

1. $L(\emptyset) = \emptyset$
2. $L(\varepsilon) = \{\varepsilon\}$
3. $L(a) = \{a\}$ for every $a \in \Sigma$
4. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
5. $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$
6. $L\left((R_1^*)\right) = (L(R_1))^*$

Example: $L(a^*b^*) = \{a^m b^n \mid m, n \geq 0\}$

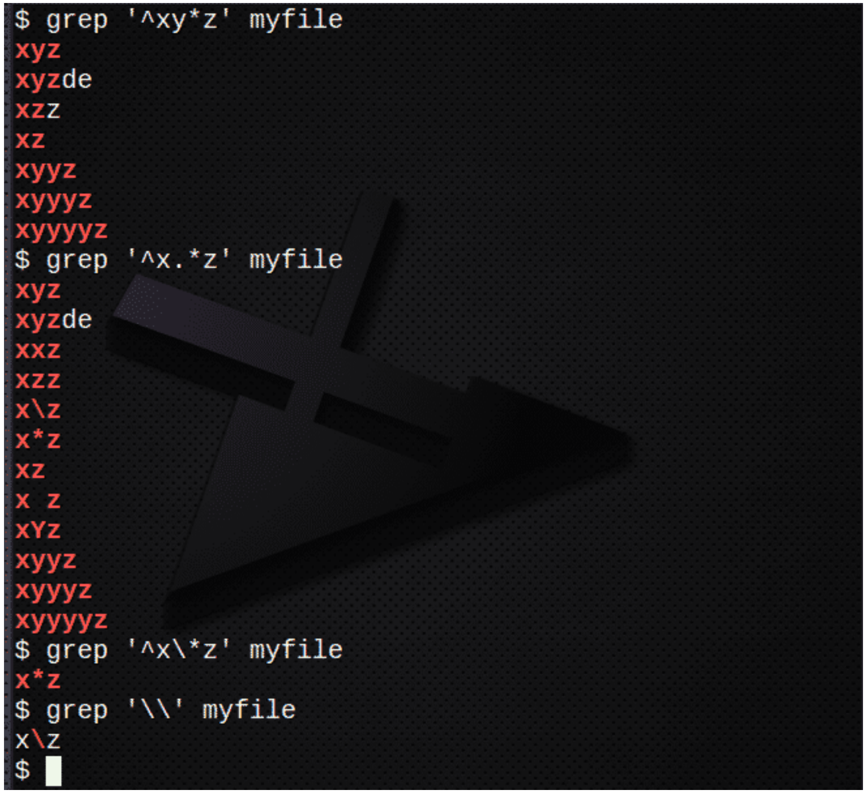
Syntactic Sugar

- For alphabet Σ, the regex Σ represents $L(\Sigma) = \Sigma$

- For regex R, the regex $R^{+} = RR^{*}$

Regexes in the Real World

grep = globally search for a regular expression and print matching lines



```
$ grep '^xy*z' myfile
xyz
xyzde
xzz
xz
xyyz
xyyyz
xyyyyz
$ grep '^x.*z' myfile
xyz
xyzde
xxz
xzz
x\z
x*z
xz
x z
xYz
xyyz
xyyyz
xyyyyz
$ grep '^x\*z' myfile
x*z
$ grep '\\' myfile
x\z
$ 
```

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

Regular expression -> NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Base cases:

$R = \emptyset$

$R = \varepsilon$

$R = a$

Regular expression → NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

What should the inductive hypothesis be?

a) Suppose **some** regular expression of length k can be converted to an NFA

b) Suppose **every** regular expression of length k can be converted to an NFA

c) Suppose **every** regular expression of length **at most** k can be converted to an NFA

d) None of the above

Regular expression → NFA

Theorem 1: Every regex has an equivalent NFA

Proof: Induction on size of a regex

Inductive step:

$$R = (R_1 \cup R_2)$$

$$R = (R_1 R_2)$$

$$R = (R_1^*)$$

Example

Convert $(1(0 \cup 1))^*$ to an NFA

Regular Expressions Describe Regular Languages

Theorem: A language A is regular if and only if it is described by a regular expression

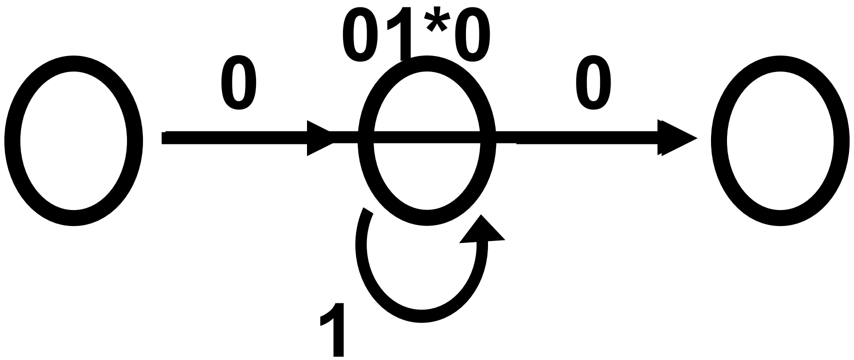
Theorem 1: Every regular expression has an equivalent NFA

Theorem 2: Every NFA has an equivalent regular expression

NFA → Regular expression

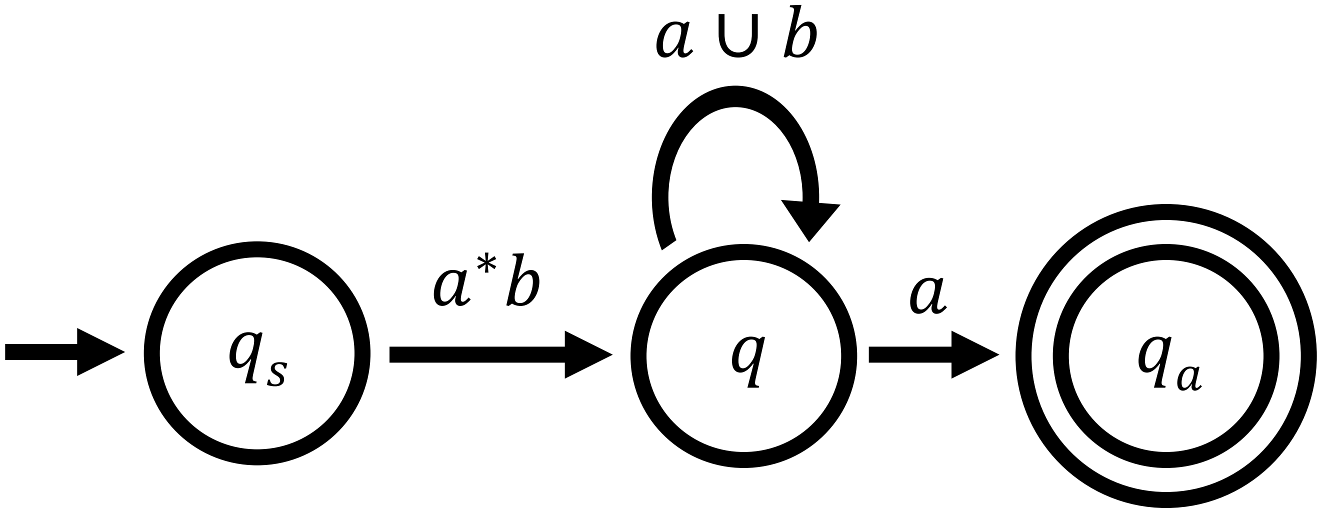
Theorem 2: Every NFA has an equivalent regex

Proof idea: Simplify NFA by “ripping out” states one at a time and replacing with regexes

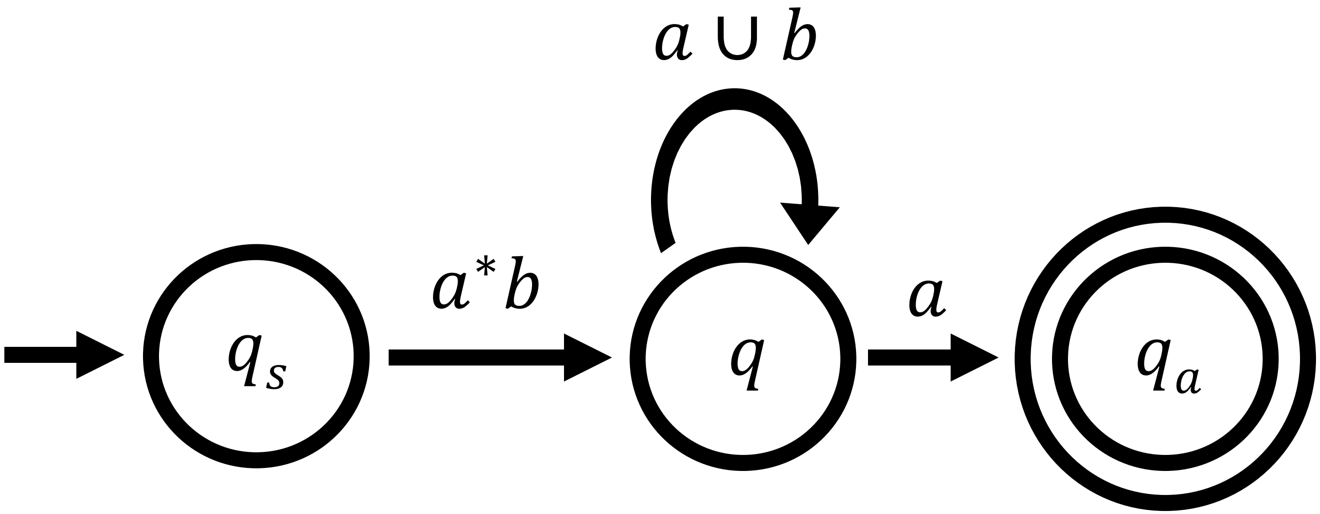


Generalized NFAs

- **Every transition is labeled by a regex**
- One start state with only outgoing transitions
- Only one accept state with only incoming transitions
- Start state and accept state are distinct



Generalized NFA Example



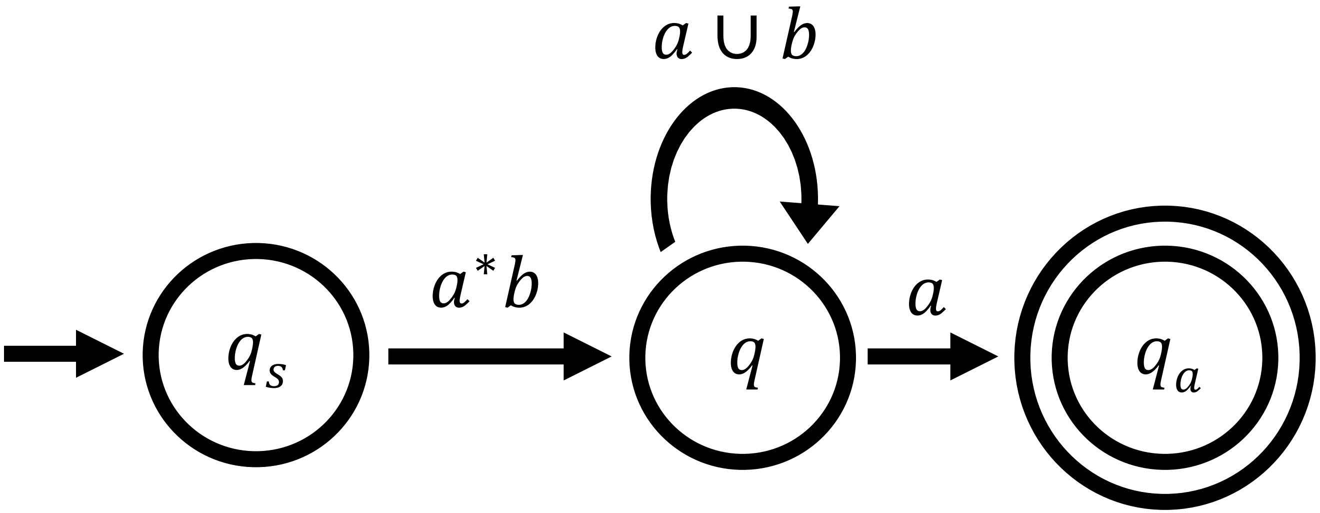
$$R(q_s, q) =$$

$$R(q_a, q) =$$

$$R(q, q_s) =$$

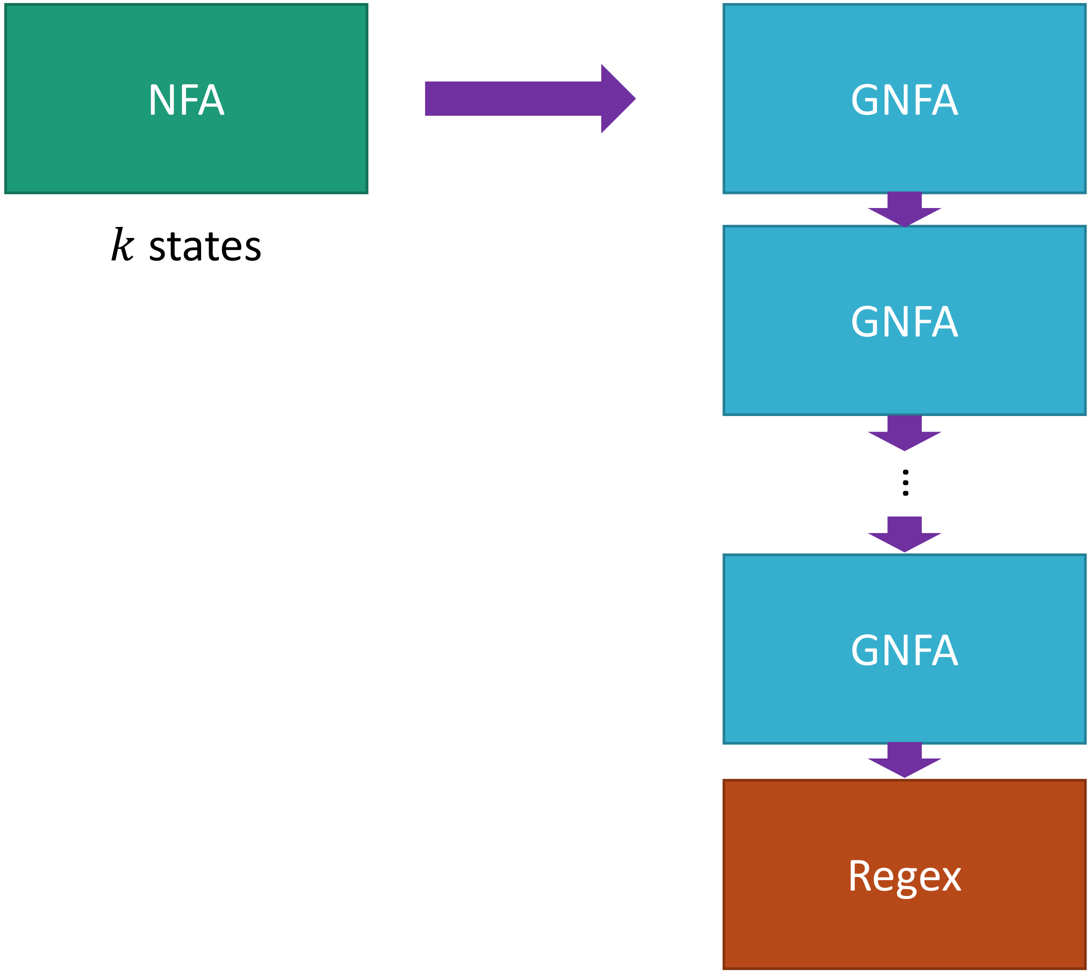
Which of these strings is accepted?

Which of the following strings is accepted by this GNFA?



a) aaa
b) $aabb$
c) bbb
d) bba

NFA → Regular expression



NFA

k states

GNFA

$k + 2$ states

GNFA

$k + 1$ states

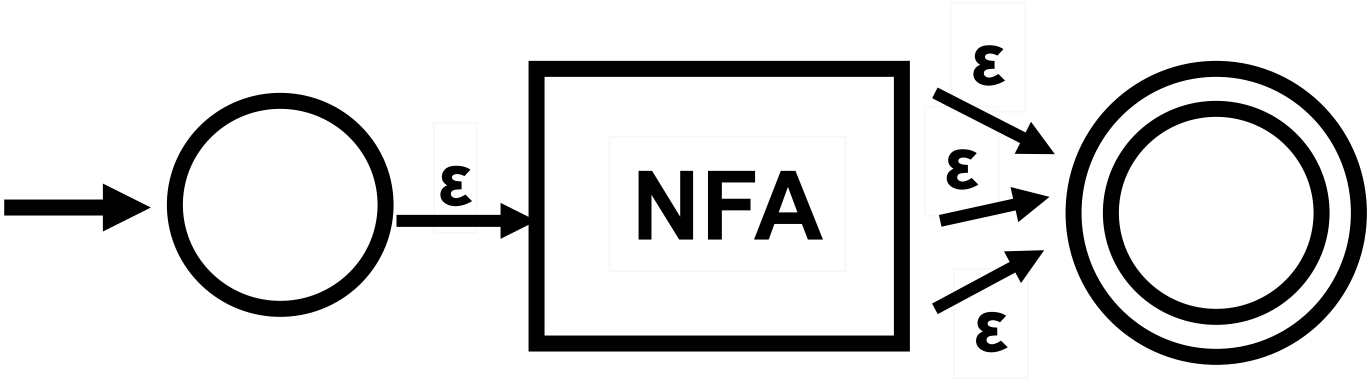
⋮

GNFA

2 states

Regex

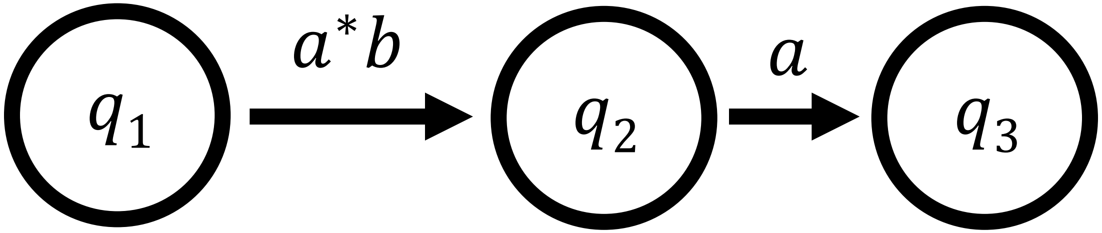
NFA → GNFA

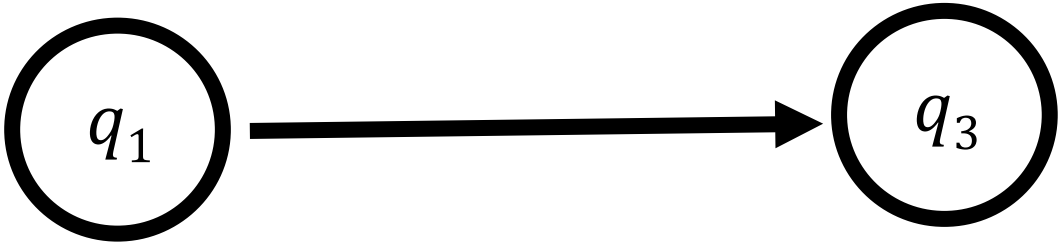


- Add a new start state with no incoming arrows.
- Make a unique accept state with no outgoing arrows.

GNFA → Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state





GNFA → Regular expression

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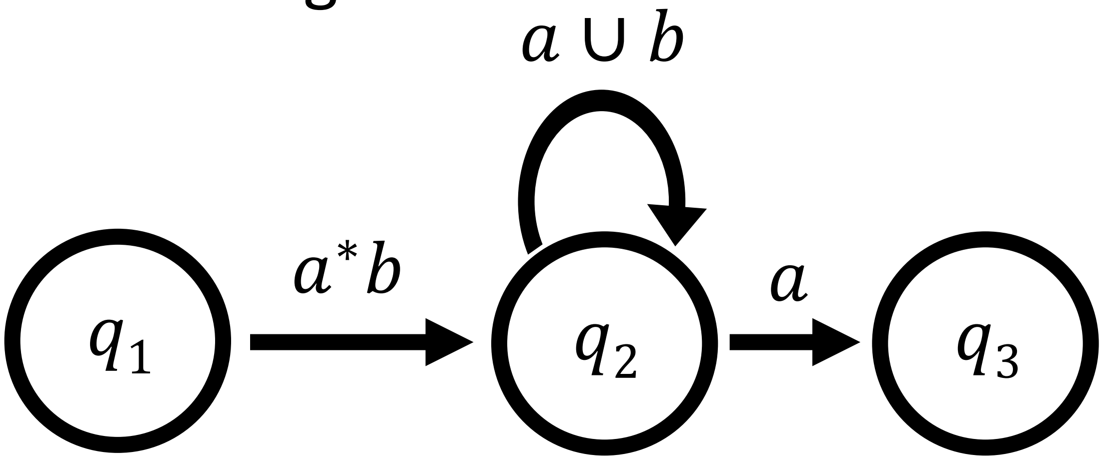
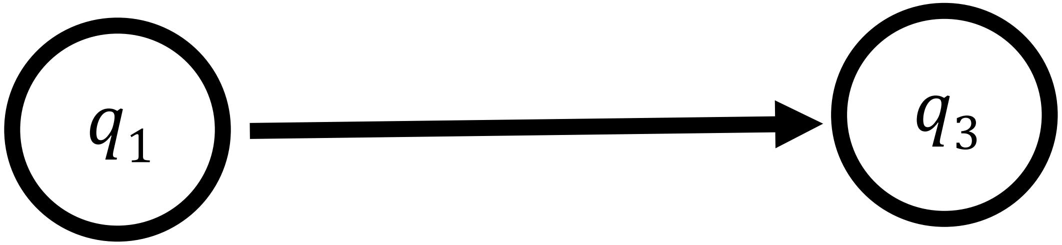


a) $a^*b(a \cup b)a$
b) $a^*b(a \cup b)^*a$
c) $a^*b \cup (a \cup b) \cup a$
d) None of the above



GNFA → Regular expression

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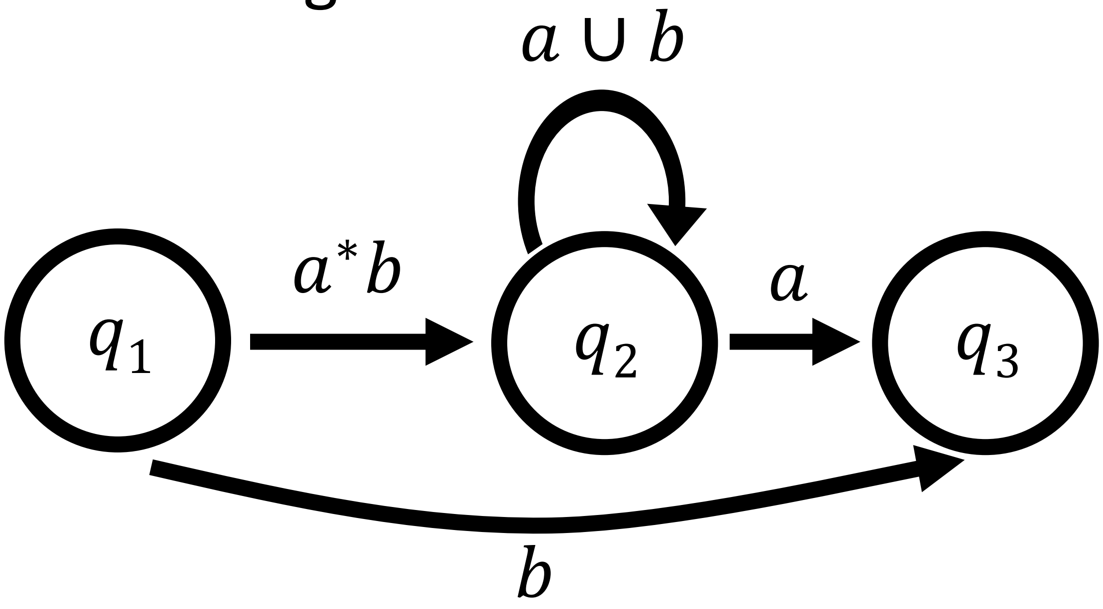
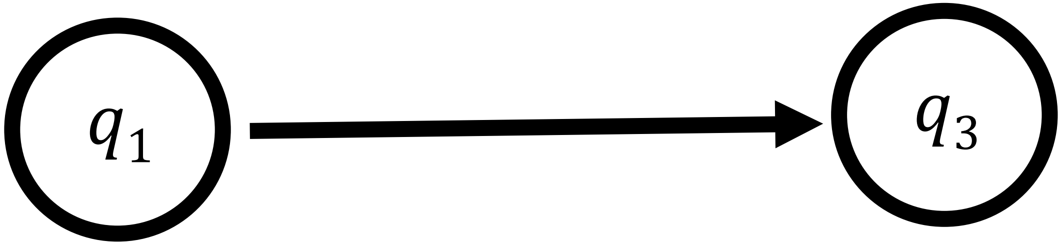
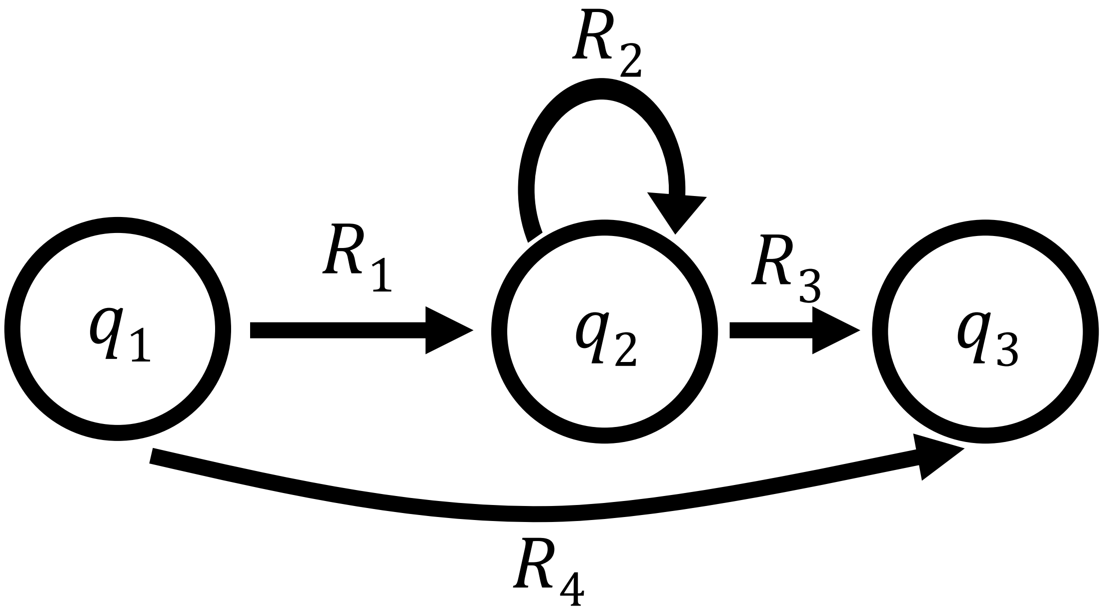
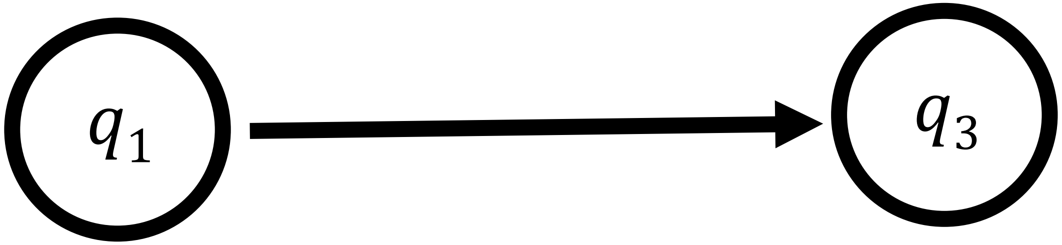


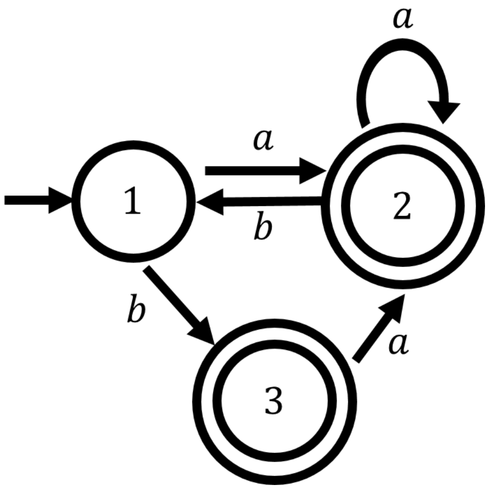


GNFA -> Regular expression

Idea: While the machine has more than 2 states, rip one out and relabel the arrows with regexes to account for the missing state





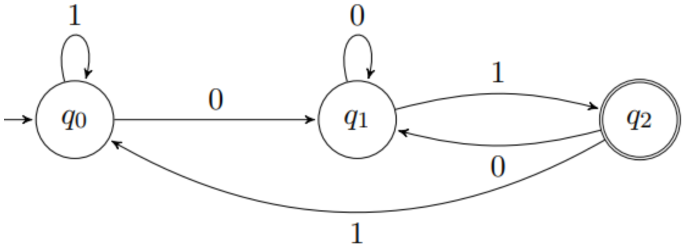

a
a
1
2
b
b
3
a

Limitations of Finite Automata

Motivating Questions

- We’ve seen techniques for showing that languages are regular
- How can we tell if we’ve found the smallest DFA recognizing a language?
- Are all languages regular? How can we prove that a language is not regular?

An Example



$A = \{\, w \in \{0,1\}^* \mid w \text{ ends with } 01 \,\}$

Claim: *Every* DFA recognizing A needs at least 3 states

Proof: Let M be any DFA recognizing A. Consider running M on each of $x = \varepsilon, y = 0, w = 01$

A General Technique

$A = \{ w \in \{0, 1\}^* \mid w \text{ ends with } 01 \}$

Definition: Strings x and y are **distinguishable** by L if there exists a "distinguishing extension" $z \in \Sigma^*$ such that exactly one of xz or yz is in L.

Ex. $x = \varepsilon, \quad y = 0$

Definition: A set of strings S is **pairwise distinguishable** by L if every pair of distinct strings $x, y \in S$ is distinguishable by L.

Ex. $S = \{\varepsilon, 0, 01\}$

A General Technique

Theorem: If S is pairwise distinguishable by L, then every DFA recognizing L needs at least $|S|$ states

Proof: Let M be a DFA with $< |S|$ states. By the pigeonhole principle, there are $x, y \in S$ such that M ends up in same state on x and y

Back to Our Example

$A = \{\, w \in \{0,1\}^* \mid w \text{ ends with } 01 \,\}$

Theorem: If S is pairwise distinguishable by L, then every DFA recognizing L needs at least $|S|$ states

$S = \{\varepsilon, 0, 01\}$