BU CS 332 – Theory of Computation

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Lecture 10:

- Turing Machines
- TM Variants and Closure Properties

Reading:

Sipser Ch 3.1-3.3

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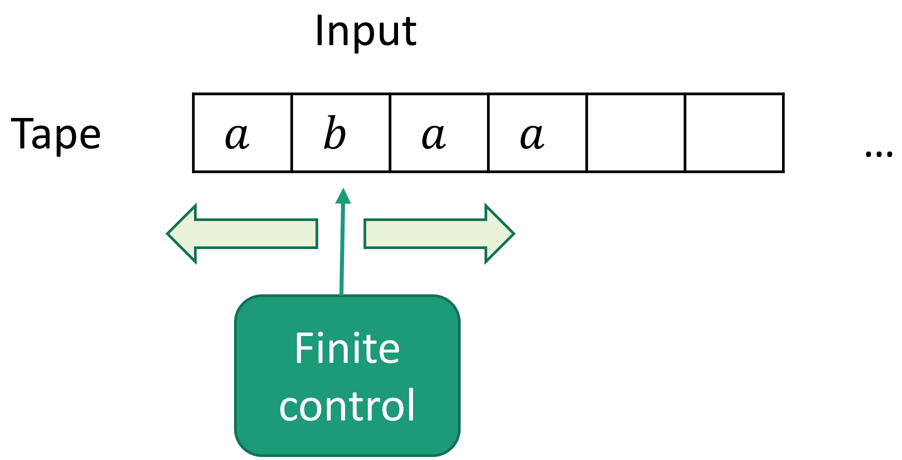
The Basic Turing Machine (TM)



- Input is written on an infinitely long tape
- Head can both read and write, and move in both directions
- Computation halts as soon as control reaches "accept" or "reject" state

Three Levels of Abstraction

High-Level Description

An algorithm (like CS 330)

Implementation-Level Description

Describe (in English) the instructions for a TM

- How to move the head
- What to write on the tape

Low-Level Description

State diagram or formal specification

Example

Determine if a string $w \in \{0\}^*$ is in the language

$$A = \{0^{2^n} \mid n \geq 0\}$$

High-Level Description

Repeat the following forever:

- If there is exactly one 0 in w, accept
- If there is an odd (> 1) number of 0s in w, reject
- Delete half of the 0s in w

Example

Determine if a string $w \in \{0\}^*$ is in the language

$$A = \{0^{2^n} \mid n \geq 0\}$$

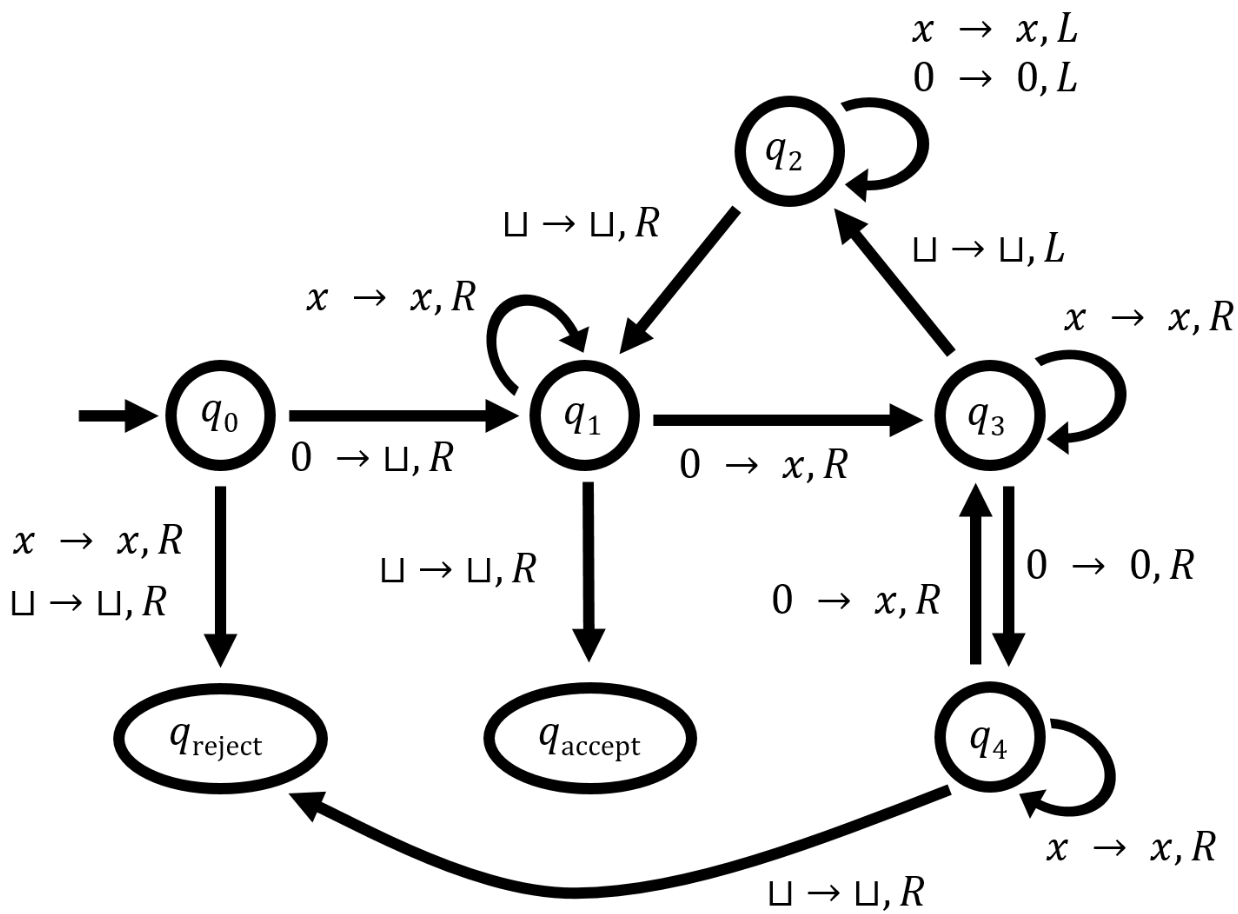
Implementation-Level Description

1. While moving the tape head left-to-right:
   a) Cross off every other 0
   b) If there is exactly one 0 when we reach the first blank symbol, accept
   c) If there is an odd (> 1) number of 0s when we reach first blank symbol, reject
2. Return the head to the left end of the tape
3. Go back to step 1

Example

Determine if a string $w \in A = \{0^{2^n} \mid n \geq 0\}$

Low-Level Description



Differences between TMs and Finite Automata

Formal Definition of a TM

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q is a finite set of states
- Σ is the input alphabet (does **not** include ⊔)
- Γ is the tape alphabet (contains ⊔ and Σ)
- δ is the transition function

…more on this later

- $q_0 \in Q$ is the start state
- $q_{\text{accept}} \in Q$ is the accept state
- $q_{\text{reject}} \in Q$ is the reject state ($q_{\text{reject}} \neq q_{\text{accept}}$)

TM Transition Function

$$\delta : Q \times \Gamma \ \to Q \times \Gamma \ \times \{L, R\}$$

L means “move left” and R means “move right”

$\delta(p, a) = (q, b, R)$ means:

- Replace a with b in current cell
- Transition from state p to state q
- Move tape head right

$\delta(p, a) = (q, b, L)$ means:

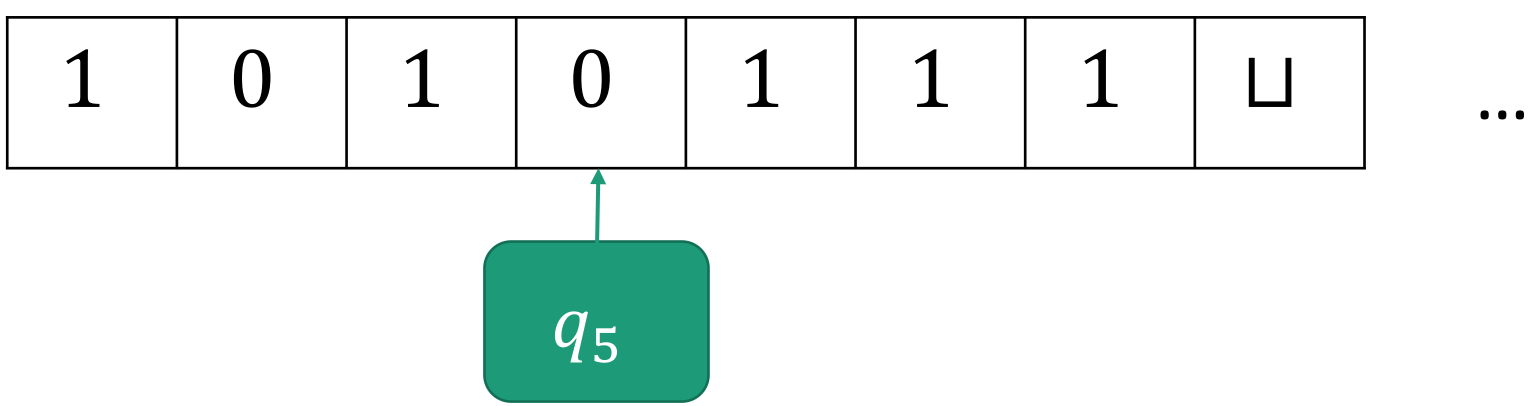
- Replace a with b in current cell
- Transition from state p to state q
- Move tape head left UNLESS we are at left end of tape, in which case don’t move

Configuration of a TM: Formally

A configuration is a string uqv where $q \in Q$ and $u, v \in \Gamma^*$

- Tape contents = uv (followed by infinitely many blanks $\sqcup$)
- Current state = q
- Tape head on first symbol of v

Example: $101q_5 0111$



How a TM Computes

Start configuration: $q_0 w$

In one step of computation:

- If $\delta(q, b) = (q', c, R)$, then $ua\ q\ bv$ yields $uac\ q'\ v$
- If $\delta(q, b) = (q', c, L)$, then $ua\ q\ bv$ yields $u\ q'\ acv$
- If we are at the left end of the tape in configuration $q\ bv$, what configuration do we reach if $\delta(q, b) = (q', c, L)$?

  a) $cq'v$
  b) $q'cv$
  c) $q' \sqcup cv$
  d) $q'cbv$

How a TM Computes

Start configuration: $q_0 w$

In one step of computation:

- If $\delta(q, b) = (q', c, R)$, then $ua\ q\ bv$ yields $uac\ q'\ v$
- If $\delta(q, b) = (q', c, L)$, then $ua\ q\ bv$ yields $u\ q'\ acv$
- If $\delta(q, b) = (q', c, L)$, then $q\ bv$ yields $q'\ cv$

Accepting configuration: $q = q_{\mathrm{accept}}$

Rejecting configuration: $q = q_{\mathrm{reject}}$

How a TM Computes

M accepts input w if there exists a sequence of configurations $C_1, \dots, C_k$ such that:

- $C_1 = q_0 w$
- C_i yields C_{i+1} for every i
- C_k is an accepting configuration

$L(M) =$ the set of all strings w which M accepts

A is Turing-recognizable if $A = L(M)$ for some TM M:

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} OR
  M runs forever on w

Recognizers vs. Deciders

$L(M)$ $=$ the set of all strings w which M accepts

A is Turing-recognizable if $A = L(M)$ for some TM M:

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} OR
  M runs forever on w

A is (Turing-)decidable if $A = L(M)$ for some TM M
which halts on every input

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject}

Recognizers vs. Deciders

Which of the following is true about the relationship between decidable and recognizable languages?

a) The decidable languages are a subset of the recognizable languages
b) The recognizable languages are a subset of the decidable languages
c) They are incomparable: There might be decidable languages which are not recognizable and vice versa

Example: Arithmetic on a TM

The following TM decides $\mathrm{MULT} = \{a^i b^j c^k \mid i \times j = k\}$:

On input string w:

1. Check w is formatted correctly
2. For each a appearing in w:
3.   For each b appearing in w:
4.   Attempt to cross off a c. If none exist, reject.
5. If all c's are crossed off, accept. Else, reject.

Example: Arithmetic on a TM

The following TM decides $\mathrm{MULT} = \{a^i b^j c^k \mid i \times j = k\}$:

On input string w:

1. Scan the input from left to right to determine whether it is a member of $L(a^* b^* c^*)$
2. Return head to left end of tape
3. Cross off an a if one exists. Scan right until a b occurs. Shuttle between b's and c's crossing off one of each until all b's are gone. Reject if all c's are gone but some b's remain.
4. Restore crossed off b's. If any a's remain, repeat step 3.
5. If all c's are crossed off, accept. Else, reject.

Back to Hilbert's Tenth Problem

Computational Problem: Given a Diophantine equation, does it have a solution over the integers?

$L =$

- L is Turing-recognizable

- L is **not** decidable (1949-70)




TM Variants

How Robust is the TM Model?

Does changing the model result in different languages being recognizable / decidable?

So far we've seen…

- Adding nondeterminism does not change the languages recognized by finite automata
- We can require that NFAs have a single accept state

Other modifications possible too: E.g., allowing DFAs to have multiple passes over their input does not increase their power

Turing machines have an astonishing level of robustness

TMs are equivalent to…

- TMs with "stay put"
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata

…

Equivalent TM models

- TMs that are allowed to "stay put" instead of moving left or right

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

TMs with stay put are *at least* as powerful as basic TMs

(Every basic TM is a TM with stay put that never stays put)

How would you show that TMs with stay put are *no more* powerful than basic TMs?

a) Convert any basic TM into an equivalent TM with stay put

b) Convert any TM with stay put into an equivalent basic TM

c) Construct a language that is recognizable by a TM with stay put, but not by any basic TM

d) Construct a language that is recognizable by a basic TM, but not by any TM with stay put

Equivalent TM models

- TMs that are allowed to "stay put" instead of moving left or right

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

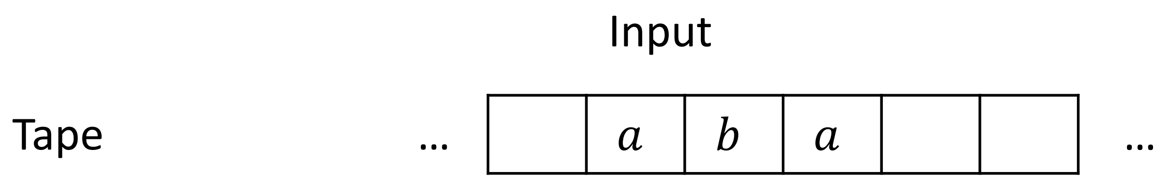
Proof that TMs with stay put are no more powerful:

Simulation: Our goal is to convert any TM M with stay put into an equivalent basic TM M'

How? Replace every stay put instruction in M with a move right instruction, followed by a move left instruction in M'

Equivalent TM models

- TMs with a 2-way infinite tape, unbounded left to right



Proof that TMs with 2-way infinite tapes are no more powerful:

Simulation: Convert any TM M with 2-way infinite tape into a 1-way infinite TM M' with a "two-track tape"

Implementation-Level Simulation

Given 2-way TM M construct a basic TM M' as follows.

TM M' = "On input $w = w_1 w_2 \dots w_n$:

1. Format 2-track tape with contents

$$\$, (w_1,\sqcup), (w_2,\sqcup), \dots, (w_n,\sqcup)$$

2. To simulate one move of M:

a) If working on upper track, read/write to the first position of cell under tape head, and move in the same direction as M

b) If working on lower track, read/write to second position of cell under tape head, and move in the opposite direction as M

c) If move results in hitting \$, switch to the other track. "

Formalizing the Simulation

Given 2-way TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, construct $M' = (Q', \Sigma, \Gamma', \delta', q_0', q_{\text{accept}}', q_{\text{reject}}')$

New tape alphabet: $\Gamma' = (\Gamma \times \Gamma) \cup \{\$\}$

New state set: $Q' = Q \times \{+, -\}$

$(q, +)$ means "in state q and working on upper track"

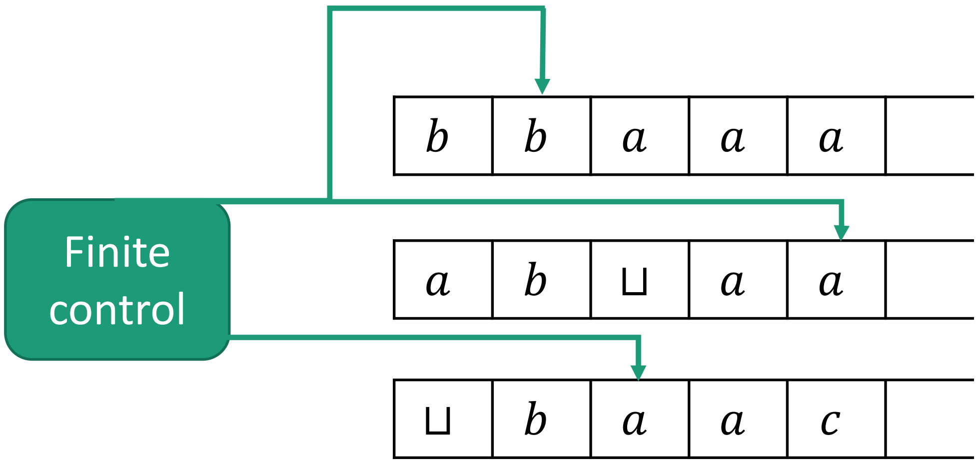
$(q, -)$ means "in state q and working on lower track"

New transitions:

If $\delta(p, a_-) = (q, b, L)$, let $\delta'\big((p, -), (a_-, a_+)\big) = ((q, -), (b, a_+), R)$

Also need new transitions for moving right, lower track, hitting $\$$, initializing input into 2-track format

Multi-Tape TMs



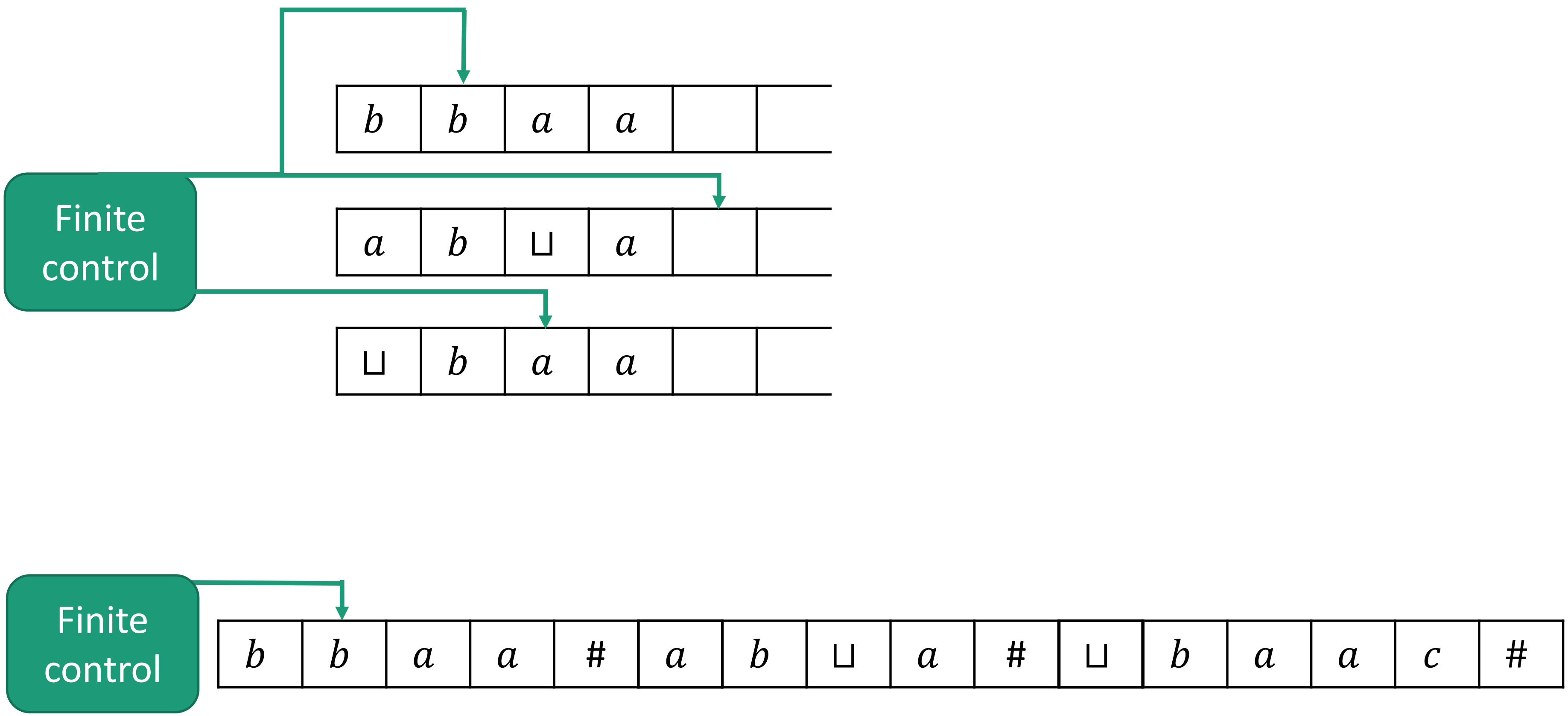
Fixed number of tapes k

(${k}$ can't depend on input or change during computation)

Transition function $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R, S\}^k$

Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k-tape TM M with can be simulated by an equivalent single-tape TM M'



Simulating Multiple Tapes

Implementation-Level Description

On input $w = w_1 w_2 \ldots w_n$

1. Format tape into $\# \dot{w}_1 w_2 \ldots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \ldots \#$
2. For each move of M:

   Scan left-to-right, finding current symbols

   Scan left-to-right, writing new symbols,

   Scan left-to-right, moving each tape head

   If a tape head goes off the right end, insert blank

   If a tape head goes off left end, move back right

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Often easier to construct multi-tape TMs

Ex. Decider for $\{a^i b^j \mid i > j\}$

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Very helpful for proving closure properties

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1, M_2 is a single-tape TM recognizing L_2

Closure Properties

The Turing-decidable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse
- Complement

The Turing-recognizable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse