

BU CS 332 – Theory of Computation

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Lecture 11:

- TM Variants
- Nondeterministic TMs
- Closure Properties

Reading:

Sipser Ch 3.2

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March 3, 2025

Last Time

Formal definition of a TM, configurations, how a TM computes

Recognizability vs. Decidability:

A is **Turing-recognizable** if there exists a TM M such that

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject} **OR**
 M runs forever on w

A is **(Turing-)decidable** if there exists a TM M such that

- $w \in A \implies M$ halts on w in state q_{accept}
- $w \notin A \implies M$ halts on w in state q_{reject}

TMs are equivalent to...

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators
- Finite automata with access to an unbounded queue
- Primitive recursive functions
- Cellular automata
- ...

Equivalent TM models

- TMs that are allowed to “stay put” instead of moving left or right

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

TMs with stay put are *at least* as powerful as basic TMs

(Every basic TM is already a TM with stay put that never stays put)

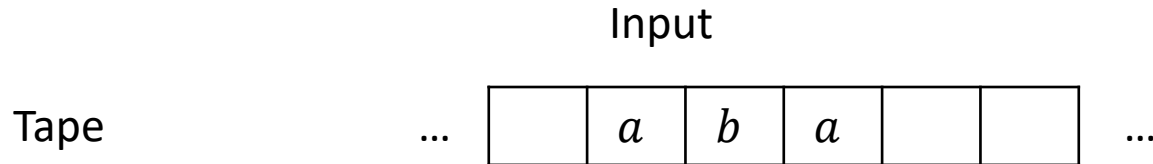
Proof that TMs with stay put are *no more* powerful:

Simulation: Our goal is to convert any TM M with stay put into an equivalent basic TM M'

How? Replace every stay put instruction in M with a move right instruction, followed by a move left instruction in M'

Equivalent TM models

- TMs with a 2-way infinite tape, unbounded left to right



Proof that TMs with 2-way infinite tapes are no more powerful:

Simulation: Convert any TM M with 2-way infinite tape into a 1-way infinite TM M' with a “two-track tape”

Implementation-Level Simulation

Given 2-way TM M construct a basic TM M' as follows.

TM M' = “On input $w = w_1w_2 \dots w_n$:

1. Format 2-track tape with contents

$\$, (w_1, \sqcup), (w_2, \sqcup), \dots, (w_n, \sqcup)$

2. To simulate one move of M :

a) If working on upper track, read/write to the first position of cell under tape head, and move in the same direction as M

b) If working on lower track, read/write to second position of cell under tape head, and move in the opposite direction as M

c) If move results in hitting $\$,$ switch to the other track. ”

Formalizing the Simulation

Given 2-way TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, construct $M' = (Q', \Sigma, \Gamma', \delta', q'_0, q'_{\text{accept}}, q'_{\text{reject}})$

New tape alphabet: $\Gamma' = (\Gamma \times \Gamma) \cup \{\$\}$

New state set: $Q' = Q \times \{+, -\}$

$(q, +)$ means “in state q and working on upper track”

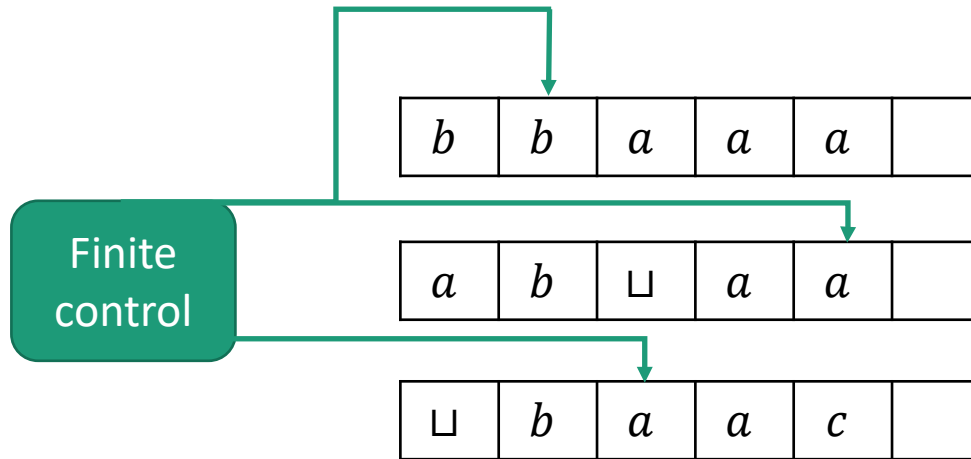
$(q, -)$ means “in state q and working on lower track”

New transitions:

If $\delta(p, a_-) = (q, b, L)$, let $\delta'((p, -), (a_-, a_+)) = ((q, -), (b, a_+), R)$

Also need new transitions for moving right, lower track, hitting \$,
initializing input into 2-track format

Multi-Tape TMs



Fixed number of tapes k

(k can't depend on input or change during computation)

Transition function $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$

Why are Multi-Tape TMs Helpful?

Theorem: Every k -tape TM M can be simulated by an equivalent single-tape TM M'

⇒ To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Often easier to construct multi-tape TMs

Ex. Decider for $\{a^i b^j \mid i > j\}$

On input w :

- 1) Scan tape 1 left-to-right to check that $w \in L(a^* b^*)$
- 2) Scan tape 2 left-to-right to copy all b 's to tape 2
- 3) Starting from left ends of tapes 1 and 2, scan both tapes to check that every b on tape 2 has an accompanying a on tape 1. If not, **reject**.
- 4) Check that the first blank on tape 2 has an accompanying a on tape 1. If so, **accept**; otherwise, **reject**.

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

Very helpful for proving **closure properties**

Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1 , M_2 is a single-tape TM recognizing L_2

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Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1 , M_2 is a single-tape TM recognizing L_2

On input w :

- 1) Scan tapes 1, 2, and 3 left-to-right to copy w to tapes 2 and 3
- 2) Repeat forever:
 - a) Run M_1 for one step on tape 2
 - b) Run M_2 for one step on tape 3
 - c) If either machine accepts, **accept**

Closure Properties

The Turing-decidable languages are closed under:

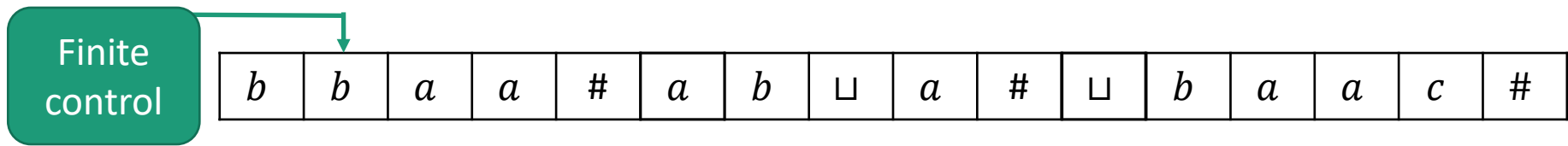
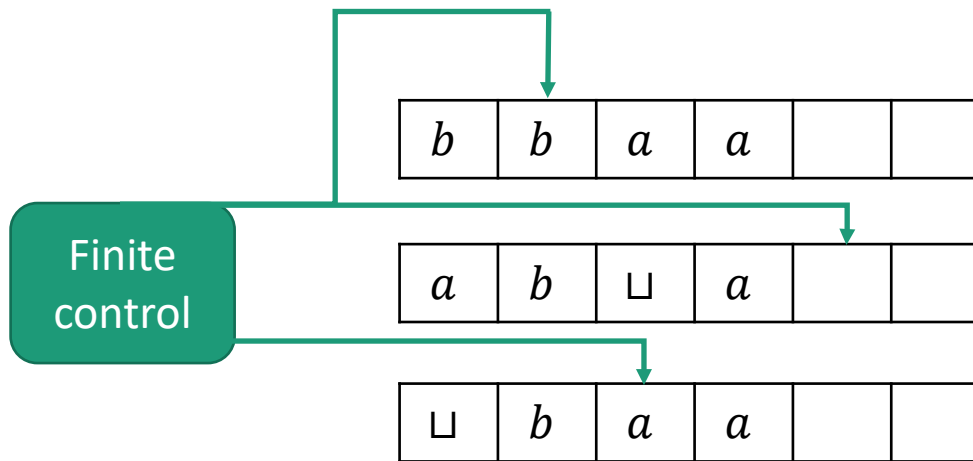
- Union
- Concatenation
- Star
- Intersection
- Reverse
- Complement

The Turing-recognizable languages are closed under:

- Union
- Concatenation
- Star
- Intersection
- Reverse

Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k -tape TM M can be simulated by an equivalent single-tape TM M'



Simulating Multiple Tapes

Implementation-Level Description

On input $w = w_1 w_2 \dots w_n$

1. Format tape into $\# w_1 w_2 \dots w_n \# \dot{\square} \# \dot{\square} \# \dots \#$

2. For each move of M :

Scan left-to-right, finding current symbols

Scan left-to-right, writing new symbols,

Scan left-to-right, moving each tape head

If a tape head goes off the right end, insert blank

If a tape head goes off left end, move back right

Why are Multi-Tape TMs Helpful?

To show a language is Turing-recognizable or decidable, it's enough to construct a multi-tape TM

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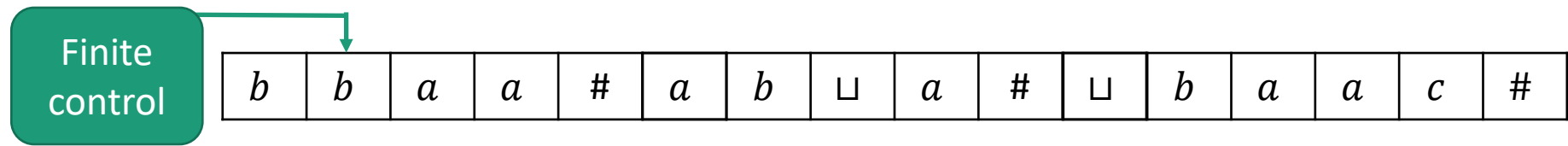
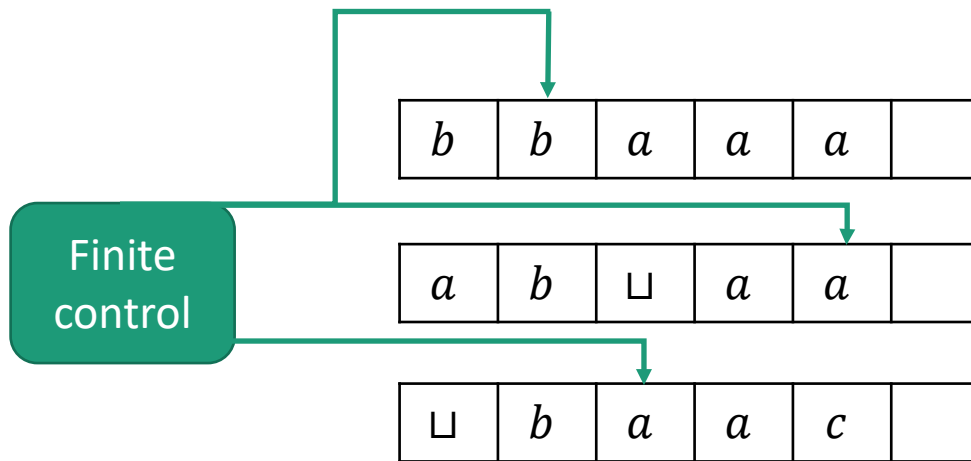
Ex. Closure of recognizable languages under union. Suppose M_1 is a single-tape TM recognizing L_1 , M_2 is a single-tape TM recognizing L_2

On input w :

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Multi-Tape TMs are Equivalent to Single-Tape TMs

Theorem: Every k -tape TM M can be simulated by an equivalent single-tape TM M'



How to Simulate It

To show that a **TM variant** is no more powerful than the **basic, single-tape TM**:

Show that if M is any variant machine, there exists a basic, single-tape TM M' that can simulate M

(Usual) parts of the simulation:

- Describe how to initialize the tapes of M' based on the input to M
- Describe how to simulate one step of M 's computation using (possibly many steps of) M'

Simulating Multiple Tapes

Implementation-Level Description of M'

On input $w = w_1 w_2 \dots w_n$

1. Format tape into $\# \dot{w}_1 w_2 \dots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \dots \#$

2. For each move of M :

Scan left-to-right, finding current symbols

Scan left-to-right, writing new symbols,

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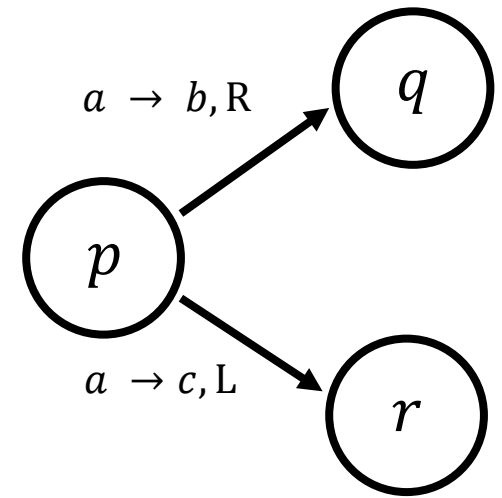
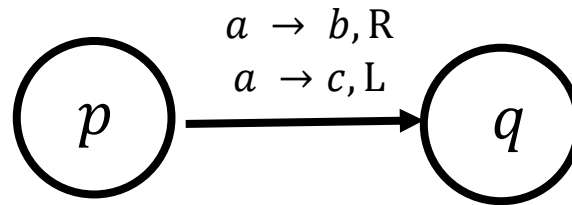
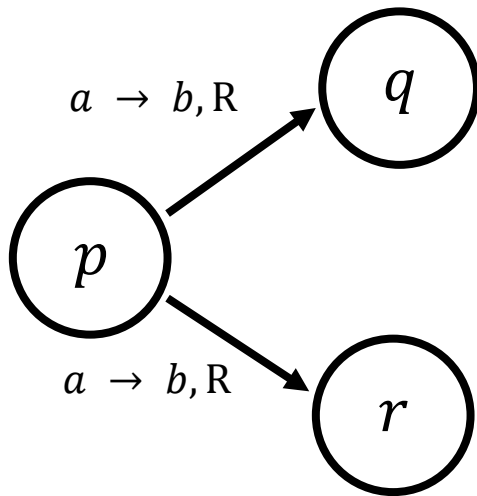
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Nondeterministic TMs

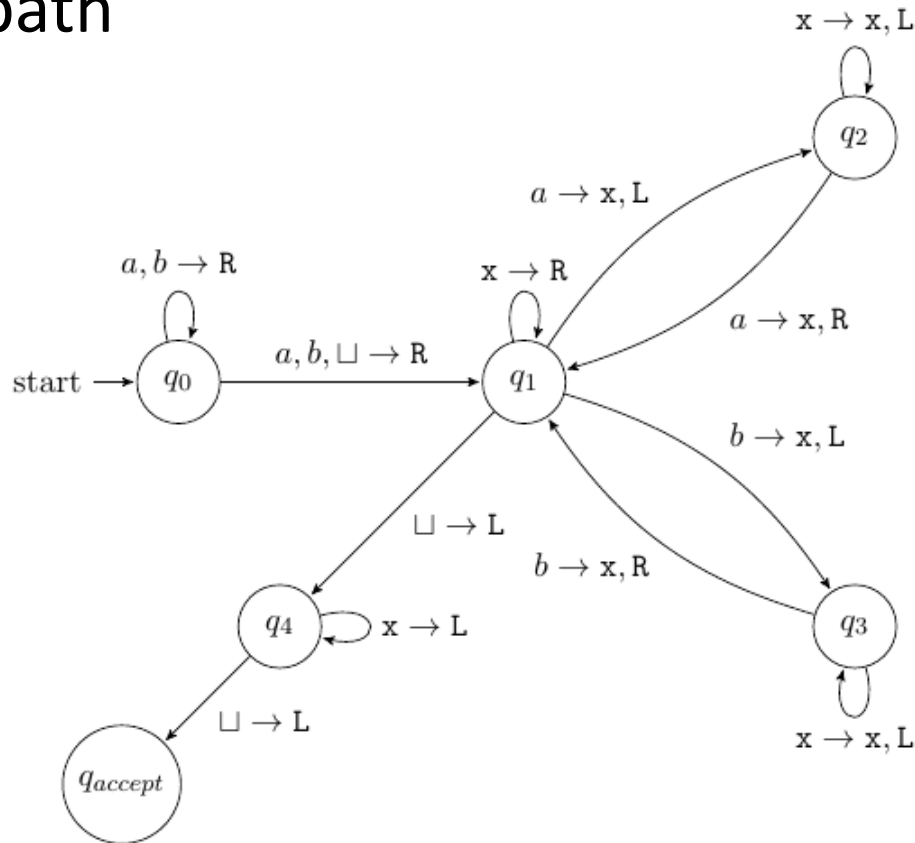
At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting branch.

Transition function $\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L, R, S\})$



Nondeterministic TMs

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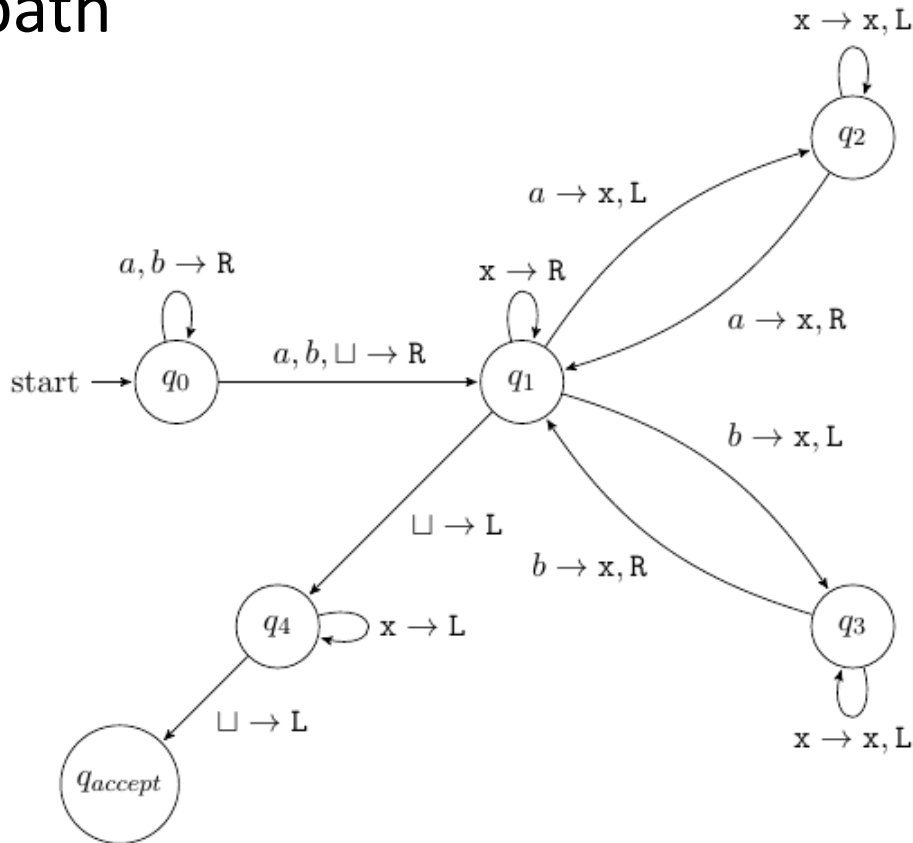
Implementation-Level Description

On input string w :

- 1) Scan tape left-to-right. At some point during this scan, nondeterministically go to step 2
- 2)
 - a) Read the next symbol s and cross it off
 - b) Move the head left repeatedly until a non- x symbol is found. If it matches s , cross it off. Else, **reject**.
 - c) Move the head right until a non- x symbol is found. If blank is hit, go to step 3.
 - d) Go back to 2a)
- 3) Check that the entire tape consists of x 's. If so, **accept**. Else, reject.

Nondeterministic TMs

At any point in computation, may nondeterministically branch. Accepts iff there exists an accepting computation path



What is the language recognized by this NTM?

- a) $\{ ww \mid w \in \{a, b\}^* \}$
- b) $\{ ww^R \mid w \in \{a, b\}^* \}$
- c) $\{ ww \mid w \in \{a, b, x\}^* \}$
- d) $\{ wx^n w^R \mid w \in \{a, b\}^*, n \geq 0 \}$

Nondeterministic TMs

An NTM N accepts input w if when run on w it accepts on at least one computational branch

$$L(N) = \{w \mid N \text{ accepts input } w\}$$

An NTM N is a decider if on **every** input, it halts on **every** computational branch

Nondeterministic TMs

Ex. Given TMs M_1 and M_2 , construct an NTM recognizing $L(M_1) \cup L(M_2)$

Nondeterministic TMs

Ex. NTM for $L = \{w \mid w \text{ is a binary number representing the product of two integers } a, b \geq 2\}$

High-Level Description:

Nondeterministic TMs

Theorem: Every nondeterministic TM can be simulated by an equivalent deterministic TM

Proof idea: Explore “tree of possible computations”

Simulating NTMs



Which of the following algorithms is always appropriate for searching the tree of possible computations for an accepting configuration?

- a) Depth-first search: Explore as far as possible down each branch before backtracking
- b) Breadth-first search: Explore all configurations at depth 1, then all configurations at depth 2, etc.
- c) Both algorithms will always work

Nondeterministic TMs

Theorem: Every nondeterministic TM has an equivalent deterministic TM

Proof idea: Simulate an NTM N using a 3-tape TM
(See Sipser for full description)

