BU CS 332 – Theory of Computation

https://forms.gle/1Gr9hdWCUw12UKdg7

Lecture 13:

- More decidable languages
- Universal Turing Machine
- Countability

Reading:

Sipser Ch 4.1, 4.2

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March 17, 2025

Last Time

Church-Turing Thesis

v1: The basic TM (and all equivalent models) capture our intuitive notion of algorithms

v2: Any physically realizable model of computation can be simulated by the basic TM

Decidable languages (from language theory)

$A_{\mathrm{DFA}} = \{\langle D, w\rangle \mid \mathrm{DFA}\ D \text{ accepts input } w\}$, etc.

Today: More decidable languages

Are there undecidable languages? How can we prove so?

A “universal” algorithm for recognizing regular languages

$A_{\text{DFA}} = \{\langle D, w \rangle \mid \text{DFA } D \text{ accepts } w\}$

Theorem: A_{DFA} is decidable

Proof: Define a (high-level) 3-tape TM M on input $\langle D, w \rangle$:

1. Check if $\langle D, w \rangle$ is a valid encoding (reject if not)
2. Simulate D on w, i.e.,
   - Tape 2: Maintain w and head location of D
   - Tape 3: Maintain state of D, update according to δ
3. Accept if D ends in an accept state, reject otherwise

Regular Languages are Decidable

Theorem: Every regular language L is decidable

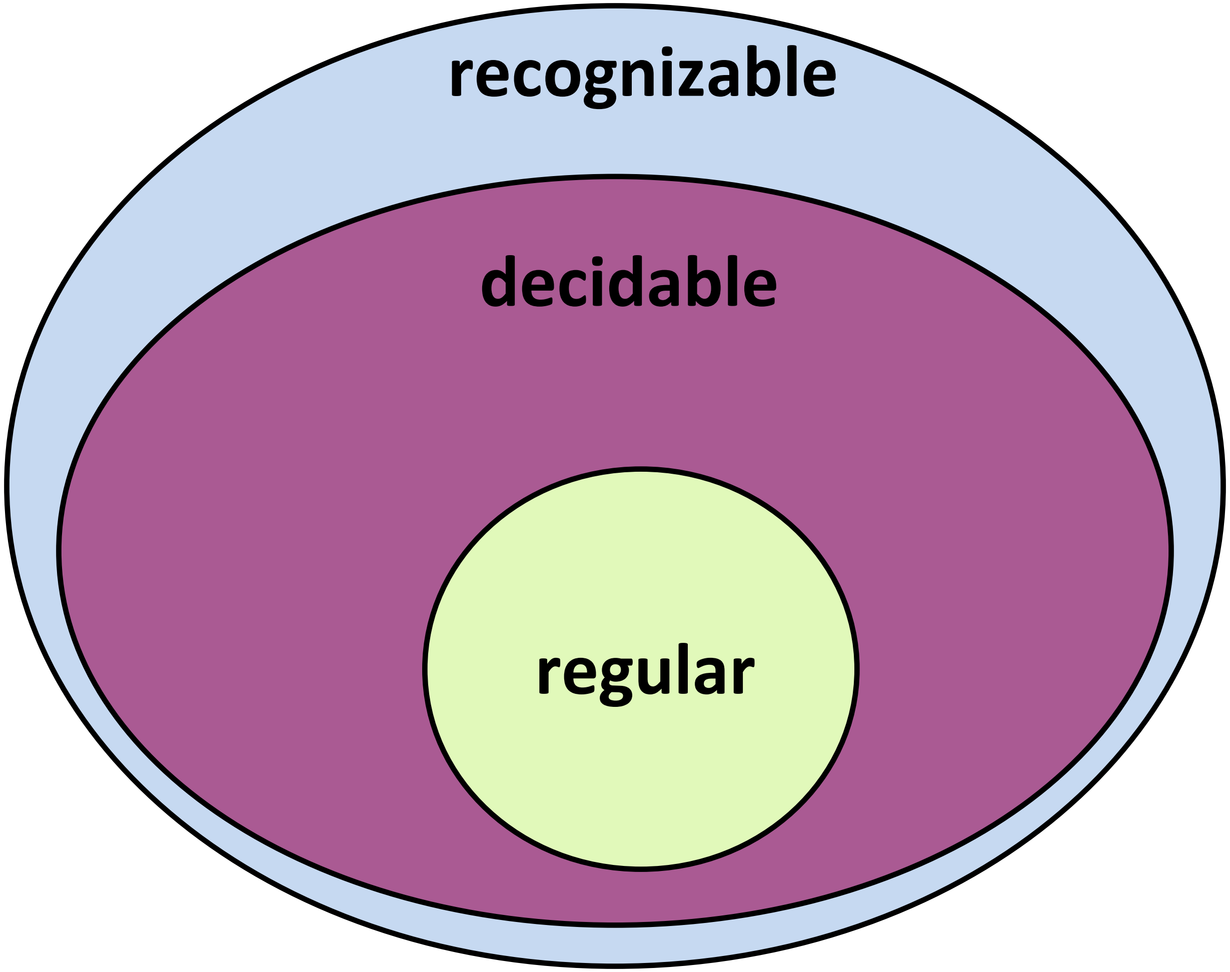
Proof 1: If L is regular, it is recognized by a DFA D. Convert this DFA to a TM M. Then M decides L.

Proof 2: If L is regular, it is recognized by a DFA D. The following TM M_D decides L.

On input w:

1. Run the decider for A_{DFA} on input $\langle D, w \rangle$
2. Accept if the decider accepts; reject otherwise

Classes of Languages



More Decidable Languages: Emptiness Testing

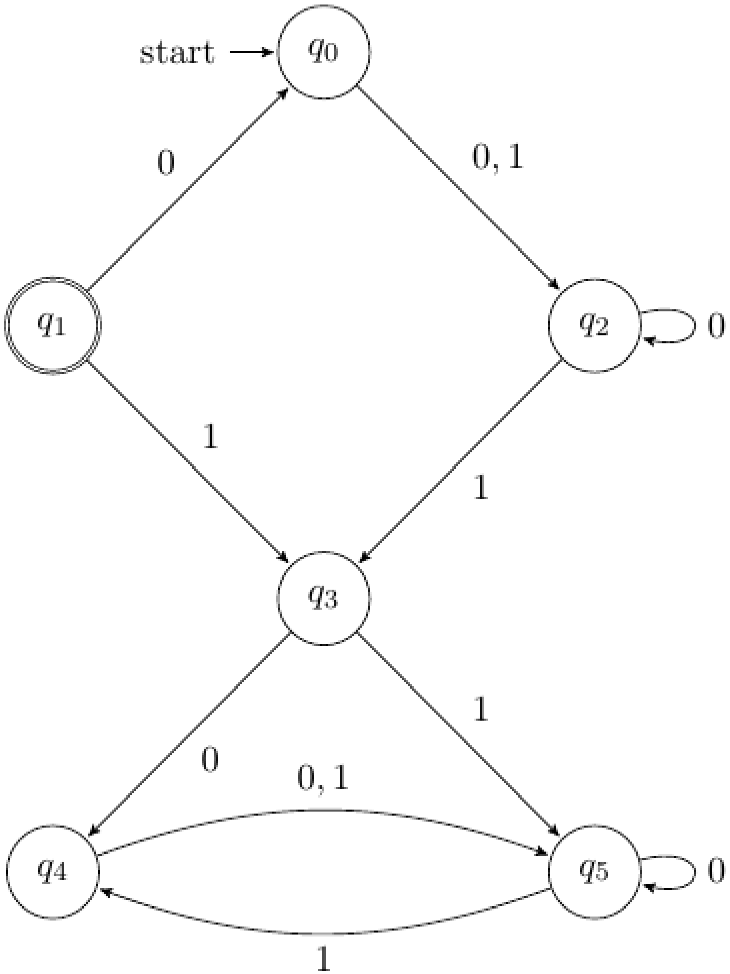
Theorem: $E_{\mathrm{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA such that } L(D) = \emptyset\}$ is decidable

Proof: The following TM decides E_{DFA}

On input $\langle D \rangle$, where D is a DFA with k states:

1. Perform k steps of breadth-first search on state diagram of D to determine if an accept state is reachable from the start state
2. Reject if a DFA accept state is reachable; accept otherwise

E_{DFA} Example



New Deciders from Old: Equality Testing

$EQ_{\mathrm{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct DFA D recognizing the **symmetric difference** $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Symmetric Difference

$$A \triangle B = \{w \mid w \in A \text{ or } w \in B \text{ but not both}\}$$

Universal Turing Machine

Meta-Computational Languages

$A_{\text{DFA}} = \{\langle D, w \rangle \mid \text{DFA } D \text{ accepts } w\}$

$A_{\text{TM}} = \{\langle M, w \rangle \mid \text{TM } M \text{ accepts } w\}$

$E_{\text{DFA}} = \{\langle D \rangle \mid \text{DFA } D \text{ recognizes the empty language } \emptyset\}$

$E_{\text{TM}} = \{\langle M \rangle \mid \text{TM } M \text{ recognizes the empty language } \emptyset\}$

$EQ_{\text{DFA}} = \{\langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs}, L(D_1) = L(D_2)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs}, L(M_1) = L(M_2)\}$

The Universal Turing Machine

$A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is Turing-recognizable

The following "Universal TM" U recognizes A_{TM}

On input $\langle M, w \rangle$:

1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

Universal TM and A_{TM}

Why is the Universal TM **not** a decider for A_{TM}?

The following "Universal TM" U recognizes A_{TM}

On input $\langle M, w \rangle$:
1. Simulate running M on input w
2. If M accepts, accept. If M rejects, reject.

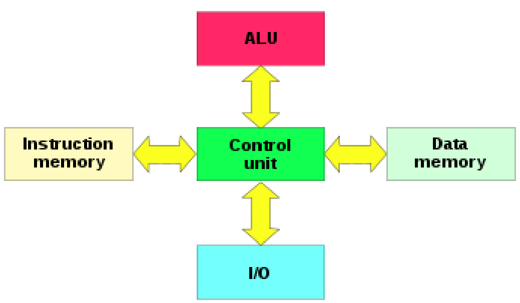
a) It may reject inputs $\langle M, w \rangle$ where M accepts w
b) It may accept inputs $\langle M, w \rangle$ where M rejects w
c) It may loop on inputs $\langle M, w \rangle$ where M loops on w
d) It may loop on inputs $\langle M, w \rangle$ where M accepts w

More on the Universal TM

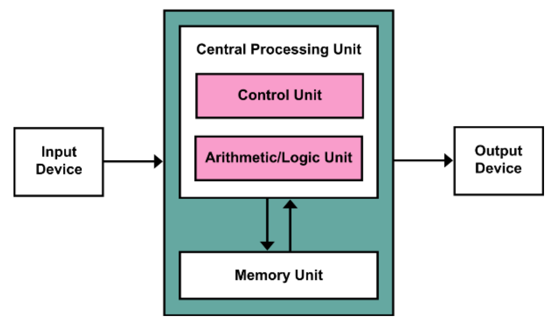
"It is possible to invent a single machine which can be used to compute any computable sequence. If this machine **U** is supplied with a tape on the beginning of which is written the S.D ["standard description"] of some computing machine **M**, then **U** will compute the same sequence as **M**."

- Turing, "On Computable Numbers…" 1936

- Foreshadowed general-purpose programmable computers
- No need for specialized hardware: Virtual machines as software



Harvard architecture:
Separate instruction and data pathways



von Neumann architecture:
Programs can be treated as data

Undecidability

A_{TM} is Turing-recognizable via the Universal TM

…but it turns out A_{TM} (and $E_{\text{TM}}, EQ_{\text{TM}}$) is **undecidable**

i.e., computers cannot solve these problems no matter how much time they are given

How can we prove this?

… but first, a math interlude

Countability and Diagonalization

What's your intuition?

Which of the following sets is the "biggest"?

a) The natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

b) The even numbers: $E = \{2, 4, 6, \ldots\}$

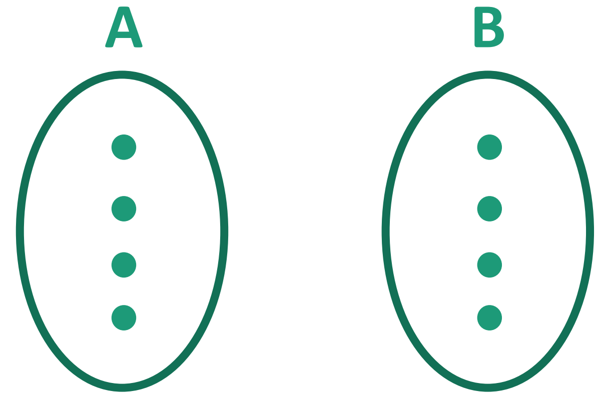
c) The positive powers of 2: $POW2 = \{2, 4, 8, 16, \ldots\}$

d) They all have the same size

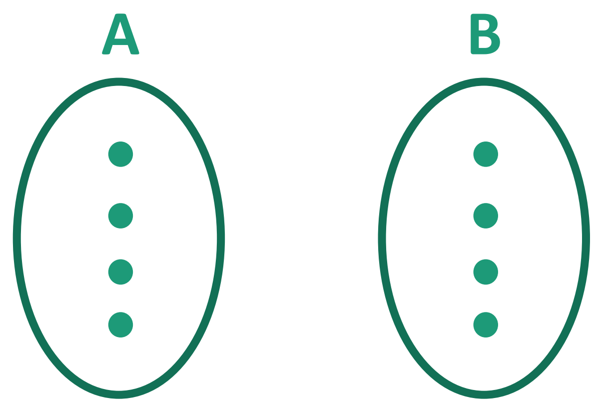
Set Theory Review

A function $f : A \to B$ is

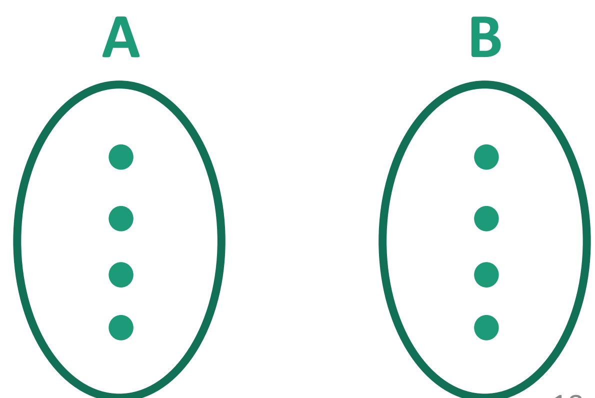
- 1-to-1 (injective) if $f(a) \neq f(a')$ for all $a \neq a'$



- onto (surjective) if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$



- a correspondence (bijective) if it is 1-to-1 and onto, i.e., every $b \in B$ has a unique $a \in A$ with $f(a) = b$



How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a bijection between them

A set is countable if either

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

Examples of countable sets

- $\emptyset$
- $\{0,1\}$
- $\{0, 1, 2, \ldots, 8675309\}$

- $E = \{2, 4, 6, 8, \ldots\}$
- $SQUARES = \{1, 4, 9, 16, 25, \ldots\}$
- $POW2 = \{2, 4, 8, 16, 32, \ldots\}$

$$|E| = |SQUARES| = |POW2| = |\mathbb{N}|$$

How to show that $\mathbb{N} \times \mathbb{N}$ is countable?

$$\begin{matrix} (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \dots \\ (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \dots \\ (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \dots \\ (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \dots \\ (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \\ & & & & & \ddots \end{matrix}$$

How to argue that a set S is countable

- Describe how to "list" the elements of S, usually in stages:

Ex: Stage 1) List all pairs (x, y) such that $x + y = 2$

Stage 2) List all pairs (x, y) such that $x + y = 3$

…

Stage n) List all pairs (x, y) such that $x + y = n + 1$

…

- Explain why every element of S appears in the list

Ex: Any $(x, y) \in \mathbb{N} \times \mathbb{N}$ will be listed in stage $x + y - 1$

- Define the bijection $f: \mathbb{N} \to S$ by $f(n) =$ the n'th element in this list (ignoring duplicates if needed)

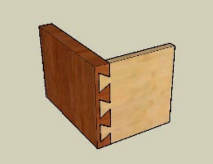
More examples of countable sets

- $\{0,1\}^*$
- $\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
- $\mathbb{Q} = \{\text{rational numbers}\}$

- If $A \subseteq B$ and B is countable, then A is countable
- If A and B are countable, then $A \times B$ is countable

- Nonempty S is countable if and only if there exists a surjection (an onto function) $f : \mathbb{N} \to S$

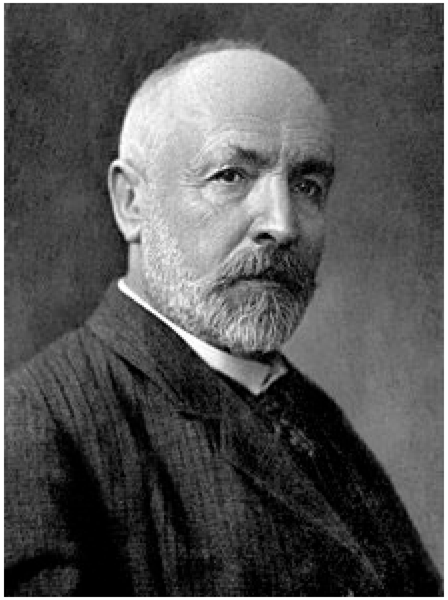
Another version of the dovetailing trick



Ex: Show that $\mathcal{F} = \{L \subseteq \{0,1\}^* \mid L \text{ is finite}\}$ is countable

So what *isn't* countable?

Cantor’s Diagonalization Method



Georg Cantor 1845-1918

- Invented set theory
- Defined countability, uncountability, cardinal and ordinal numbers, …

Some praise for his work:

“Scientific charlatan…renegade…corruptor of youth”
–L. Kronecker

“Set theory is wrong…utter nonsense…laughable”
–L. Wittgenstein

Uncountability of the reals

Theorem: The real interval $[0, 1]$ is uncountable.

Proof: Assume for the sake of contradiction it were countable, and let $f: \mathbb{N} \to [0,1]$ be a surjection

n	$f(n)$
1	$0\,.\,d_1^1\, d_2^1\, d_3^1\, d_4^1\, d_5^1 \ldots$
2	$0\,.\,d_1^2\, d_2^2\, d_3^2\, d_4^2\, d_5^2 \ldots$
3	$0\,.\,d_1^3\, d_2^3\, d_3^3\, d_4^3\, d_5^3 \ldots$
4	$0\,.\,d_1^4\, d_2^4\, d_3^4\, d_4^4\, d_5^4 \ldots$
5	$0\,.\,d_1^5\, d_2^5\, d_3^5\, d_4^5\, d_5^5 \ldots$

Construct $b \in [0,1]$ which does not appear in this table

– contradiction!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

Diagonalization

This process of constructing a counterexample by "contradicting the diagonal" is called diagonalization

Structure of a diagonalization proof

Say you want to show that a set T is uncountable

1) Assume, for the sake of contradiction, that T is countable with surjection $f: \mathbb{N} \rightarrow T$
2) "Flip the diagonal" to construct an element $b \in T$ such that $f(n) \neq b$ for every n

   Ex: Let $b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$
   (where d_n^n is digit n of $f(n)$)

3) Conclude (by contradiction) that f is not a surjection

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a surjection $f: X \to P(X)$

<u>What should we do?</u>

a) Show that for every $S \in P(X)$, there exists $x \in X$ such that $f(x) = S$

b) Construct a set $S \in P(X)$ (meaning, $S \subseteq X$) that cannot be the output $f(x)$ for any $x \in X$

c) Construct a set $S \in P(X)$ and two distinct $x, x' \in X$ such that $f(x) = f(x') = S$

Diagonalization argument

Assume a surjection $f: X \to P(X)$

x					
x_1					
x_2					
x_3					
x_4					
$\vdots$					

Diagonalization argument

Assume a surjection $f: X \to P(X)$

x	$x_1 \in f(x)$?	$x_2 \in f(x)$?	$x_3 \in f(x)$?	$x_4 \in f(x)$?	…
x_1	Y	N	Y	Y	
x_2	N	N	Y	Y	
x_3	Y	Y	Y	N	
x_4	N	N	Y	N	
$\vdots$					$\ddots$

Define S by flipping the diagonal:

Put $x_i \in S \iff x_i \notin f(x_i)$

Example

Let $X = \{1, 2, 3\}$, $P(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Ex. $f(1) = \{1, 2\}$, $f(2) = \emptyset$, $f(3) = \{2\}$

x	$1 \in f(x)$?	$2 \in f(x)$?	$3 \in f(x)$?
1			
2			
3			

Construct $S =$

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a surjection $f: X \to P(X)$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$