Homework 1 – Due Thursday, January 29, 2026 at 11:59 PM

Problems There are 8 required problems and one bonus problem.

1. (**Review of Sets and Functions**) For a natural number n, let $[n]$ denote the set $\{1, 2, 3, \ldots, n-1, n\}$.
   (a) What is $[n] \cap [n+1]$?
   (b) What is $[n] \cup [n+1]$?
   (c) Is there an surjective (onto) function from $[n+1]$ to $[n]$? Prove your answer.
   (d) Is there an injective (one-to-one) function from $[n+1]$ to $[n]$? Prove your answer.

2. (**Review of Logic**) Consider the following statement: "If it is rush hour, then every MBTA subway car has at least ten passengers."
   (a) What is the negation of this statement?
   (b) What is the converse of this statement?
   (c) What is the contrapositive of this statement?

3. For each of the following languages, (i) give a plain English description of the language, (ii) describe the decision problem corresponding to that language (i.e., "Given a string ... determine whether ..."), and (iii) give two examples of strings in the language and two examples of strings that are in Σ^* but are outside the language.
   (a) $L_1 = \{0x1 \mid x \in \{0,1\}^*\}$ where the alphabet $\Sigma = \{0, 1\}$
   (b) $L_2 = \{w \in \{a,b\}^* \mid |w| \geq 5\} \cap \overline{\{x\mathsf{abba}y \mid x, y \in \{\mathsf{a},\mathsf{b}\}^*\}}$ where the alphabet $\Sigma = \{a, b\}$

4. For each of the following languages, (i) describe the language using set-builder notation and union/intersection/complement/reverse/concatenation operations (the notation used in Problem 3), (ii) describe the decision problem corresponding to that language, and (iii) give two examples of strings in the language and two examples of strings that are in Σ^* but are outside the language.
   (a) L_3 = the set of all strings over alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ that are palindromes (read the same forwards and backwards) or that end with c. (For example, bacab and cabbac are palindromes.)
   (b) L_4 = the set of all strings over alphabet $\{1, 2, \ldots, 9\}$ whose length is divisible by s, where s is the first symbol of the string.

5. Which of the following statements are true or false, for all alphabets Σ? For each, provide either a proof or a counterexample.
   (a) **INDIVIDUAL:** For all strings $x, y, z \in \Sigma^*$, we have $|x^R \circ y \circ \varepsilon \circ z| = |x| + |y| + |z|$. (Recall that $\circ$ denotes string concatenation.)
   (b) For all languages $L_1, L_2 \subseteq \Sigma^*$, we have $(L_1 \cap L_2)^R = L_1^R \cap L_2^R$.
   (c) For every *finite* language $L \subseteq \Sigma^*$, we have $|L \circ L| = |L|^2$.

(d) For all languages $L_1, L_2, L_3 \subseteq \Sigma^*$, we have $L_1 \cup (L_2 \circ L_3) = (L_1 \cup L_2) \circ (L_1 \cup L_3)$.

6. Let L be a language consisting of strings over an alphabet Σ. Define

$$\text{Prefix}(L) = \{x \in \Sigma^* \mid xy \in L \text{ for some } y \in \Sigma^*\}.$$

That is, Prefix(L) consists of all (possibly empty) strings which appear as prefixes of strings in L.

(a) Let $A = \{00, 001, 1011\}$. What is Prefix(A)?

(b) Let $B = \{\mathsf{ab}^n \mid n \geq 0\}$. What is Prefix($B$)?

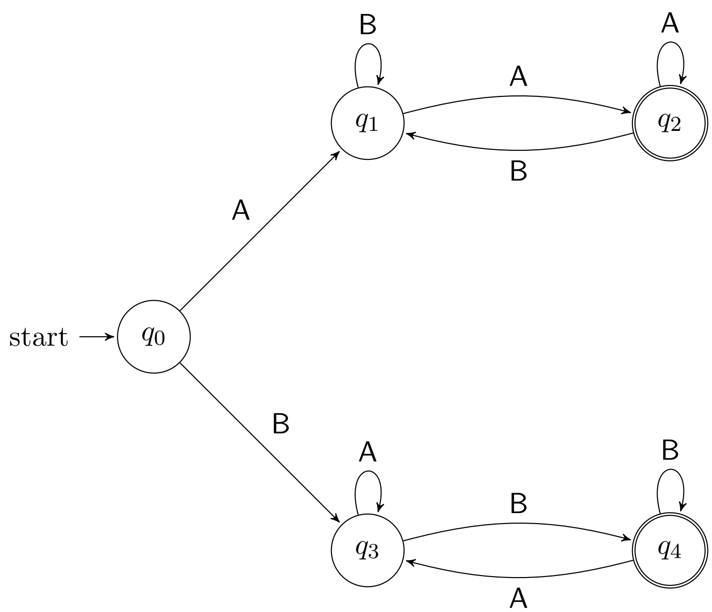
(c) Prove that if L is a finite language, then Prefix(L) is also finite.

7. Let $G = (V, E)$ be a finite directed graph. A *walk* in G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that every consecutive pair v_i, v_{i+1} is connected by an edge in E.

(a) A *directed cycle* in G is a walk of the form $s, v_1, \ldots, v_{k-1}, s$ where the vertices $v_1, \ldots, v_{k-1}$ are all distinct. Prove that if G has a directed cycle, then there are infinitely many walks in G.

(b) Prove the converse to part (a): If there are infinitely many walks in G, then G has a directed cycle.

8. Consider the following state diagram of a DFA M using alphabet $\Sigma = \{\mathsf{A}, \mathsf{B}\}$.



(a) What is the start state of M?

(b) What is the set of accept states of M?

(c) Give a formal description of the machine M (i.e., as a 5-tuple).

(d) What sequence of states does the machine go through on input ABBAB?

(e) Does the machine accept the string ABBAA?

(f) Give a simple description of language recognized by M. (Either plain English or set-builder notation is fine.)

9. **Bonus Problem.** In this problem, you'll explore how to formally define and analyze properties of strings. Let Σ be an alphabet. A *string* over alphabet Σ is defined recursively as follows. It is either the empty string ε (base case) or takes the form ax where $a \in \Sigma$ and x is itself a string (recursive case). The *length* of the string x can then be defined as follows:

$$|x| = \begin{cases} 0 & \text{if } x = \varepsilon \\ 1 + |z| & \text{if } x = az \text{ for } a \in \Sigma \text{ and } z \in \Sigma^*. \end{cases}$$

Similarly, the concatenation of two strings x, y can be defined as

$$xy = \begin{cases} y & \text{if } x = \varepsilon \\ a(zy) & \text{if } x = az \text{ for } a \in \Sigma \text{ and } z \in \Sigma^*. \end{cases}$$

These definitions let us give inductive proofs of properties of strings. For instance, consider the following claim, which says that the length of the concatenation of two strings is the sum of the lengths of those strings.

Claim. For any two strings x, y, we have $|xy| = |x| + |y|$.

To prove this, let x and y be arbitrary strings. We will prove this by induction on the length $n = |x|$. As our base case, suppose $n = 0$. Then $x = \varepsilon$, so

$$\begin{aligned} |xy| &= |\varepsilon y| && \text{(assumption on } x) \\ &= |y| && \text{(definition of concatenation)} \\ &= |\varepsilon| + |y| && \text{(definition of length)} \\ &= |x| + |y| && \text{(assumption on } x) \end{aligned}$$

as we wanted. Now assume as our inductive hypothesis that the claim is true for length n; we want to show it is true for length $n + 1$. In this case, we have $x = az$ for some string z of length n. So

$$\begin{aligned} |xy| &= |a(zy)| && \text{(definition of concatenation)} \\ &= 1 + |zy| && \text{(definition of length)} \\ &= 1 + |z| + |y| && \text{(inductive hypothesis)} \\ &= |x| + |y| && \text{(definition of length).} \end{aligned}$$

(a) Given a string $x \in \{0,1\}^*$ and a character $b \in \{0,1\}$, let $\text{num}(x, b)$ denote the number of times b appears in x. For example, $\text{num}(10110, 0) = 2$. Give a recursive definition of the num function along the lines of what we did with length above.

(b) Given strings $x, y \in \{0,1\}^*$ and a character $b \in \{0,1\}$, let $\text{map}(x, y, b)$ denote the string obtained by replacing every instance of the character b in x with the string y. For example, $\text{map}(10110, 101, 0) = 110111101$. Give a recursive definition of the map function.

(c) Give an inductive proof that $|\,\text{map}(x, y, b)| = |y| \cdot \text{num}(x, b) + \text{num}(x, 1 - b)$ for all strings $x, y \in \{0,1\}^*$ and $b \in \{0,1\}$.