Homework 8 – Due Thursday, April 2 at 11:59 PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators and write "Collaborators: none" if you worked by yourself. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden. Collaboration is not allowed on problems marked "INDIVIDUAL."

Note You may use various generalizations of the Turing machine model we have seen in class, such as TMs with two-way infinite tapes, stay-put, or multiple tapes. If you choose to use such a generalization, state clearly and precisely what model you are using. **You may describe Turing machines at a high-level on this assignment.**

There are 4 required problems and 2 optional bonus problems on this assignment.

1. (**Ugly TMs**) A two-tape Turing machine M on input alphabet $\Sigma = \{\mathsf{a}, \mathsf{b}, \ldots, \mathsf{z}\}$ is *ugly* if there exists a string $w \in \Sigma^*$ such that, on input w, the TM M ever has the substring "cygnet" appear somewhere on its second tape when run on input w.[1] Consider the problem of determining whether (the encoding of) a TM M is ugly.

   (a) Formulate this problem as a language $\mathrm{UGLY}_{\mathrm{TM}}$. Caution: The only input to this computational problem is $\langle M \rangle$ for a TM M.

   (b) Prove that the language $\mathrm{UGLY}_{\mathrm{TM}}$ is undecidable.
   Hint: Give a reduction from the undecidable language A_{TM}. That is, you should assume for the sake of contradiction that $\mathrm{UGLY}_{\mathrm{TM}}$ is decidable. Then under this assumption, construct a TM deciding A_{TM}, explain why your decider is correct, and as a result conclude that your assumption that $\mathrm{UGLY}_{\mathrm{TM}}$ is decidable must have been false. It's also fine if you want to give a reduction from a different undecidable language, instead, but your proof should still have this structure.

   (c) Is $\mathrm{UGLY}_{\mathrm{TM}}$ Turing-recognizable? Is $\overline{\mathrm{UGLY}_{\mathrm{TM}}}$ Turing-recognizable? Give a convincing explanation for both of your answers, but a complete description of a TM or of a reduction is not necessary.

2. (**Subset detection**) Consider the following computational problem: Given the (encodings of) two basic TMs M and N, determine whether the language recognized by M is a subset of the language recognized by N.

   (a) Formulate this problem as a language $\mathrm{SUBSET}_{\mathrm{TM}}$.

   (b) Prove that the language $\mathrm{SUBSET}_{\mathrm{TM}}$ is undecidable.

[1] Happy International Children's Book Day on April 2! https://en.wikipedia.org/wiki/The_Ugly_Duckling

3. (**Mapping Reductions**) Prove the following general statements about the existence of mapping reductions.

   (a) Let A be an arbitrary recognizable language, and suppose $A \leq_{\mathrm{m}} \overline{A}$. Show that A is decidable. (Hint: Recall that a language is decidable iff both it and and its complement are recognizable.)

   (b) Let A be an arbitrary decidable language and let B be an arbitrary language other than $\emptyset$ or Σ^*. (That is, there exists a string $w_0 \notin B$ and a string $w_1 \in B$.) Show that $A \leq_{\mathrm{m}} B$.

4. (**Disjoint TMs**) Consider the following computational problem: Given TMs M_1 and M_2, is it the case that $L(M_1) \cap L(M_2) = \emptyset$? This problem is captured by the language $DISJ_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) \cap L(M_2) = \emptyset\}$.

   (a) If f is a mapping reduction from A_{TM} to $\overline{DISJ_{\mathrm{TM}}}$, what should the inputs and outputs to f be?

   (b) Describe a TM computing a mapping reduction from A_{TM} to $\overline{DISJ_{\mathrm{TM}}}$ and explain why this TM is correct.

   (c) Explain why part (b) implies that $DISJ_{\mathrm{TM}}$ is not Turing-recognizable.

   (d) Is there is a mapping reduction from A_{TM} to $DISJ_{\mathrm{TM}}$? Prove your answer.

5. (**Bonus problem # 1**) Define the language $\mathrm{XOR}_{\mathrm{TM}} = \{\langle M, w, v \rangle \mid M \text{ is a TM that accepts exactly one of the strings } w, v\}$. Prove that both $\mathrm{XOR}_{\mathrm{TM}}$ and its complement $\overline{\mathrm{XOR}_{\mathrm{TM}}}$ are both unrecognizable.

6. (**Bonus problem # 2**) Show that the assumption that A is recognizable in Problem 3 is necessary: Give an example of an unrecognizable language A such that $A \leq_m \overline{A}$.