BU CS 332 – Theory of Computation

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Lecture 8:

- More on non-regularity
- Test 1 review

Reading:

"Myhill-Nerode" note

HW 3 due tomorrow 11:59 PM

Test 1 next Thursday 2/19

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Last Time: Distinguishing Set Method

Definition: Strings x and y are **distinguishable** by L if there exists a "distinguishing extension" $z \in \Sigma^*$ such that exactly one of xz or yz is in L.

Definition: A set of strings S is **pairwise distinguishable** by L if every pair of distinct strings $x, y \in S$ is distinguishable by L.

Theorem: If S is pairwise distinguishable by L, then every DFA recognizing L needs at least $|S|$ states.

Corollary: If language L has an infinite pairwise distinguishable set, then L is not regular.

Palindromes

0 1 0
0 1 0 1 0

Theorem: $L = \{w \in \{0,1\}^* \mid w = w^R\}$ is not regular

Proof: We construct an infinite pairwise distinguishable set

Let $S = \{(01)^n \mid n \geq 0\}$

Let $S' = \{0^n \mid n \geq 0\}$

Let $x = 0^m$, $y = 0^n$ $m \neq n$

Let $z = 10^m$

Then $xz = 0^m 1 0^m \in L$

$yz = 0^n 1 0^m \notin L$

$(m \neq n)$

Let $x = (01)^m$

$y = (01)^n$ for $m \neq n$

Let $z = (10)^m$ $(= x^R)$

$xz = (01)^m (10)^m$

$= \underbrace{0101\ldots01}_{2m}\ \underbrace{1010\ldots10}_{2m} \in L$

$yz = \underbrace{0101\ldots01}_{2n}\ \underbrace{1010\ldots10}_{2m} \notin L$

Now you try!

Use the distinguishing set method to show that the following language is not regular

$L_1 = \{0^i 1^j \mid i > j \geq 0\}$

Your job: Build an infinite set S such that for all $x \neq y \in S$, there exists a z such that exactly one of xz and yz is in L

Attempt 1: $S = L(0^*1^*) = \{0^i 1^j \mid i,j \geq 0\}$

$x = 0^{i_1} 1^{j_1}$

$y = 0^{i_2} 1^{j_2}$?

Attempt 2: $S = \{0^n 1 \mid n \geq 1\}$

$x = 0^m 1$

$y = 0^n 1$ take $m > n$ WLOG

$z = 1^{n-1}$

$xz = 0^m 1^n \in L$

$yz = 0^n 1^n \notin L$

Perfect Squares

Use the distinguishing set method to show that the following language is not regular

$$L_2 = \{1^{n^2} \mid n \geq 0\}$$

Let $S = \{1^{n^2} \mid n \geq 0\}$

To show S is pairwise distinguishable:

Let $x = 1^{m^2}$

$y = 1^{n^2}$ WLOG $m > n$

Attempt 1: $z = 1^{m^2 - n^2}$

$xz = 1^{m^2} \circ 1^{m^2 - n^2} = 1^{2m^2 - n^2}$

Might "accidentally" also be in L

e.g. $m = 5$, $n = 1$

$\Rightarrow 2m^2 - n^2 = 49 = 7^2$

$yz = 1^{n^2} \circ 1^{m^2 - n^2} = 1^{m^2} \in L$

Perfect Squares

Use the distinguishing set method to show that the following language is not regular

$$L_2 = \{1^{n^2} \mid n \geq 0\}$$

Let $S = \{1^{n^2} \mid n \geq 0\}$

To show S is pairwise distinguishable:

Let $x = 1^{m^2}$

$y = 1^{n^2}$ WLOG $m > n$

Attempt 2: (works)

$z = 1^{2n+1}$

$xz = 1^{m^2} \circ 1^{2n+1} = 1^{m^2+2n+1} \notin L_2$

Not a perfect square

$m^2 + 2n + 1 < m^2 + 2m + 1 = (m+1)^2$

$m^2 < m^2 + 2n + 1 < (m+1)^2$

$yz = 1^{n^2} \circ 1^{2n+1} = 1^{n^2+2n+1} = 1^{(n+1)^2} \in L_2$

Reusing a Proof

Finding a distinguishing set can take some work…

Let's try to reuse that work!

How might we show that

$$BALANCED = \{w \mid w \text{ has an equal \# of 0s and 1s}\}$$

is not regular?

known non-regular | regular: $= L(0^*1^*)$

$$\{0^n1^n \mid n \geq 0\} = BALANCED \cap \{w \mid \text{all 0s in } w \text{ appear before all 1s}\}$$

Claim: BALANCED is not regular

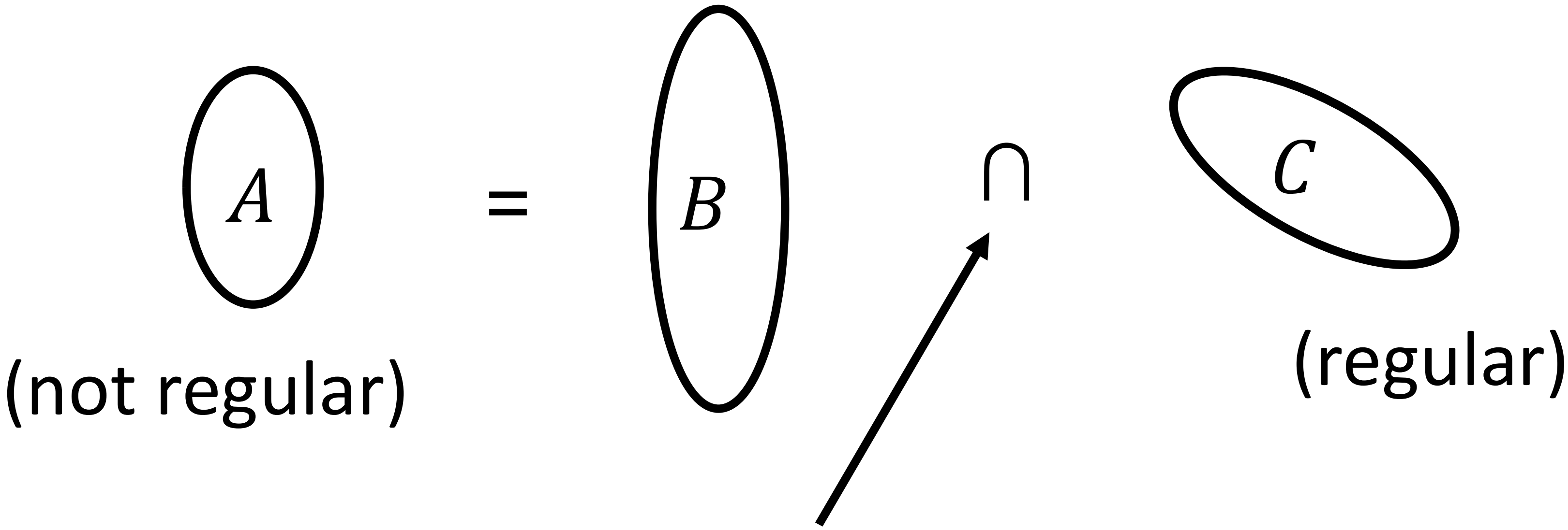
Proof. Assume FTSOC that BALANCED is regular

$\Rightarrow$ $BALANCED \cap L(0^*1^*)$ is regular (closure under intersection)

$\Rightarrow$ $\{0^n1^n \mid n \geq 0\}$ is regular ⨳

Using Closure Properties

If A is not regular, we can show a related language B is not regular



any of $\{\circ, \cup, \cap\}$ or, for one language, $\{\neg, {}^R, *\}$

<u>By contradiction:</u> If B is regular, then $B \cap C$ $(= A)$ is regular.

But A is not regular so neither is B!

Example

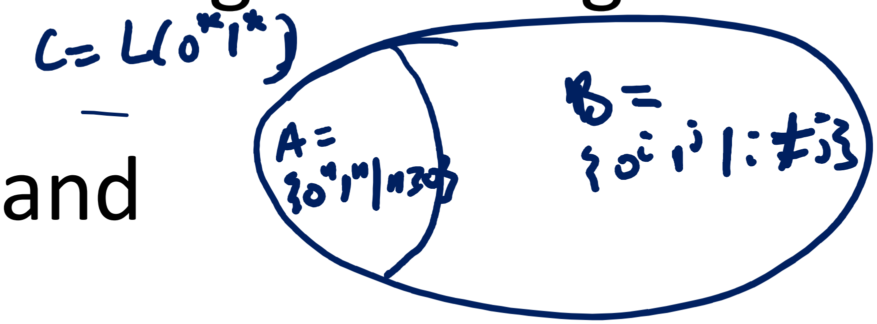
Prove $B = \{0^i 1^j | i \neq j\}$ is not regular using

- Nonregular language

  $A = \{0^n 1^n | n \geq 0\}$ and

- Regular language

  $C = \{w \mid \text{all 0s in } w \text{ appear before all 1s}\}$



Which of the following expresses A in terms of B and C?

a) $A = B \cap C$

b) $A = \bar{B} \cap C$

c) $A = B \cup C$

d) $A = \bar{B} \cup C$

Proof that B is nonregular

Assume for the sake of contradiction that B is regular

We know: $A = \bar{B} \cap C$

nonregular (A) — $L(0^* 1^*)$ is regular (C)

$\Rightarrow \bar{B}$ is regular (closure under complement)

$\Rightarrow \bar{B} \cap C$ is regular (closure under intersection)

$\Rightarrow A$ is regular ✗

!DANGER!

Let $B = \{0^i 1^j | i \neq j\}$ and write $B = A \cup C$ where

- Nonregular language

$$A = \{0^i 1^j | i > j \geq 0\} \text{ and}$$

- Nonregular language

$$C = \{0^i 1^j | j > i \geq 0\} \text{ and}$$

Does this let us conclude B is nonregular?

Not necessarily: The union of two non-regular languages could be regular

Ex: $A = \{0^i 1^j | i > j \geq 0\}$ non-regular

$\overline{A}$ also non-regular (If $\overline{A}$ were regular, then $\overline{(\overline{A})} = A$ would also be regular ✗)

But $A \cup \overline{A} = \{0,1\}^*$ is regular.

Test 1 Topics

Sets, Strings, Languages (0)

- Know the definition of a string and of a language (and the difference between them)
- Understand operations on strings: Concatenation, reverse

  $x \circ y = xy = x_1 \dots x_n y_1 \dots y_m$
- Understand operations on languages: Union, intersection, concatenation, reverse, star, complement

  $L_1 \circ L_2 = \{ xy \mid x \in L_1, y \in L_2 \}$
- Know the difference between $\emptyset$ and ε

Deterministic FAs (1.1)

- Given an English or formal description of a language L, draw the state diagram of a DFA recognizing L (and vice versa)
- Know the formal definition of a DFA (A DFA is a 5 tuple…) and convert between state diagram and formal description
- Know the formal definition of how a DFA computes
- Construction for closure of regular languages under complement

Nondeterministic FAs (1.2)

- Given an English or formal description of a language L, draw the state diagram of an NFA recognizing L (and vice versa)
- Know the formal definition of an NFA
- Know the subset construction for converting an NFA to a DFA
- Proving closure properties: Know the constructions for union, concatenation, star
- Know how to prove your own closure properties

Regular Expressions (1.3)

- Given an English or formal description of a language L, construct a regex generating L (and vice versa)
- Formal definition of a regex

  Base cases: a, ϕ, ε

  Inductive: $\cup$, $\circ$, $*$
- Know how to convert a regex to an NFA
- Know how to convert a DFA/NFA to a regex

Limitations of DFAs (Myhill-Nerode Note)

- Understand the statements of the distinguishing set method for proving DFA size lower bounds / non-regularity
- Understand the proof of why the distinguishing set method works, and be able to use it to prove similar statements
- Know how to apply the method to specific languages
- We won't ask you to prove that some language is non-regular, since you didn't have any homework problems on this yet

Test format

Problem 1: "Check your type checker"

E.g., Is aabba a string, language, or a regex?

How about {ab} U {aab}? Language

Problem 2: True/false with **justification**

Either provide a convincing explanation or a specific counterexample

Problems 3-5(?) Homework-style problems

Study tips

- Make sure you know how to solve the problems on the practice test and are familiar with the format. The format/length of the real test will be very similar.

- If you need more practice, there are lots of problems in the book. We're happy to talk about any of these problems in office hours.

- You may bring a page of notes (writing on both sides ok) to the test. Preparing this note sheet is a great study aid.

Test tips

- You may cite without proof any result…
  - Stated in lecture
  - Stated and proved in the main body of the text (Ch. 0-1.3 + Myhill-Nerode note)
  - These include worked-out examples of state diagrams, regexes
- Not included above: homework problems, discussion problems, (solved) exercises/problems in the text

- Showing your work / explaining your answers will help us give you partial credit
- Make sure you're interpreting quantifiers (for all / there exists) correctly and in the correct order

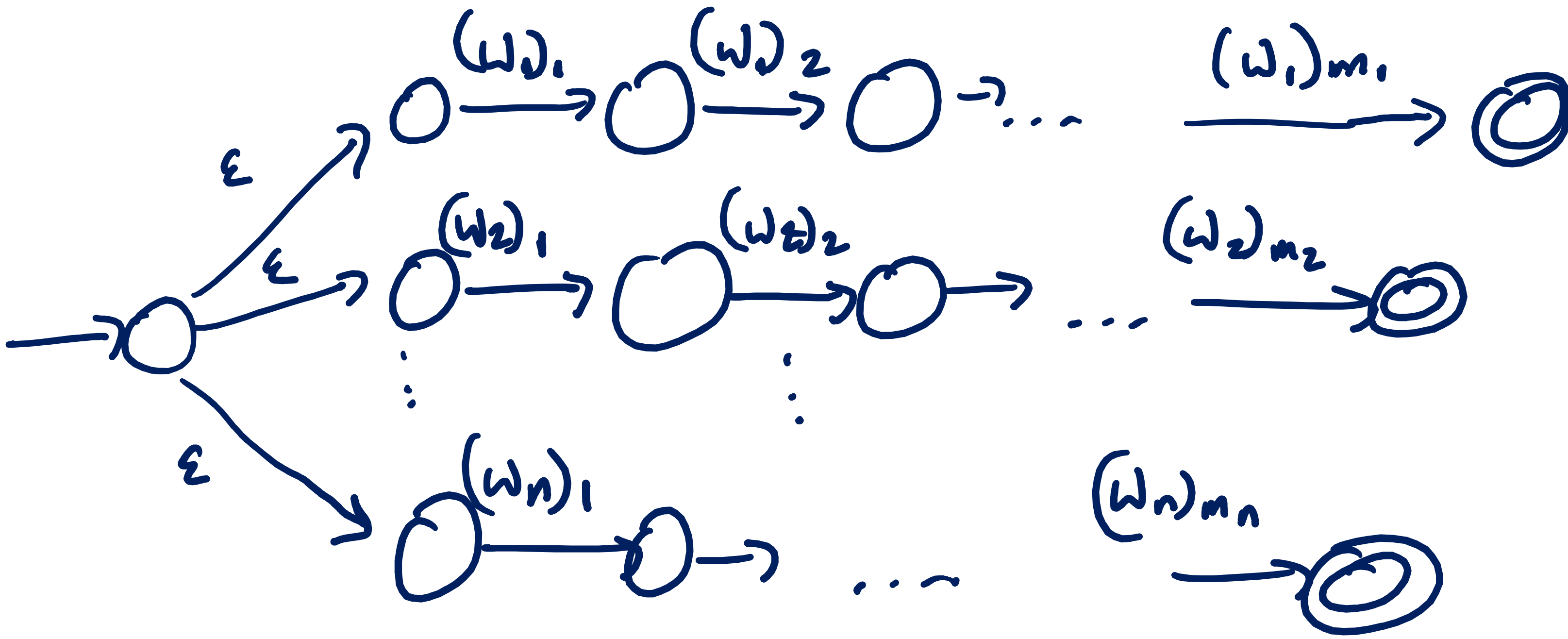
Practice Problems

Name six operations under which the regular languages are closed

Prove or disprove: All finite languages are regular

Let L be a finite language. Then $L = \{w_1, \ldots, w_n\}$ for some finite # n of strings $w_1, \ldots, w_n$.

where $m_1 = |w_1|$

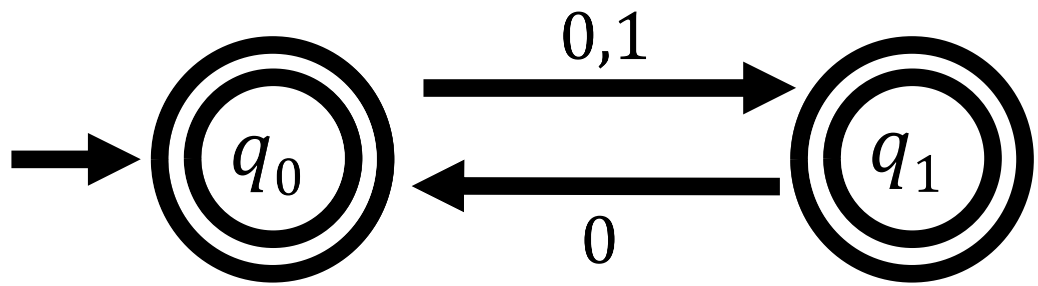


is an NFA recognizing $L \Rightarrow L$ is regular

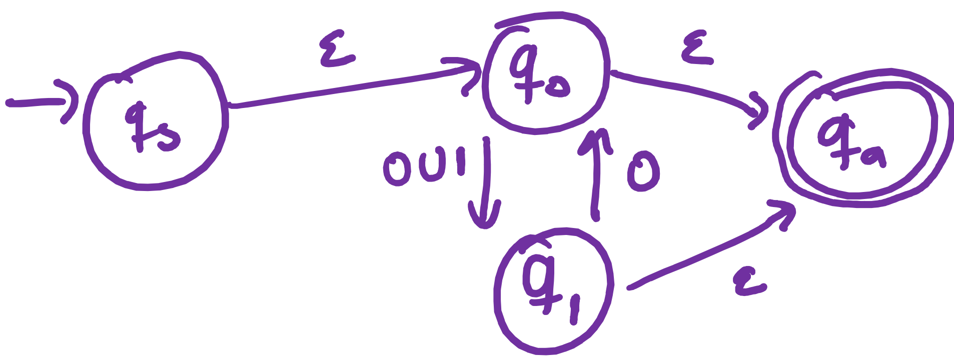
Prove or disprove: The **non-**regular languages are closed under intersection

Give the state diagram of an NFA recognizing the language $(01 \cup 10)^* \circ 1$

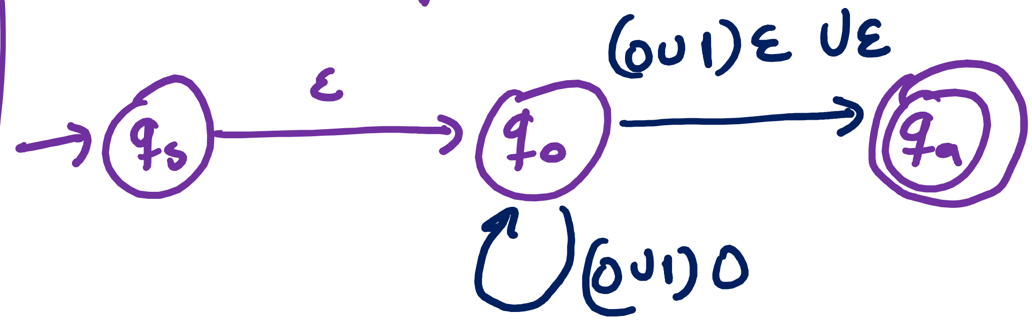
Give an equivalent regular expression for the following NFA



1) Convert to GNFA

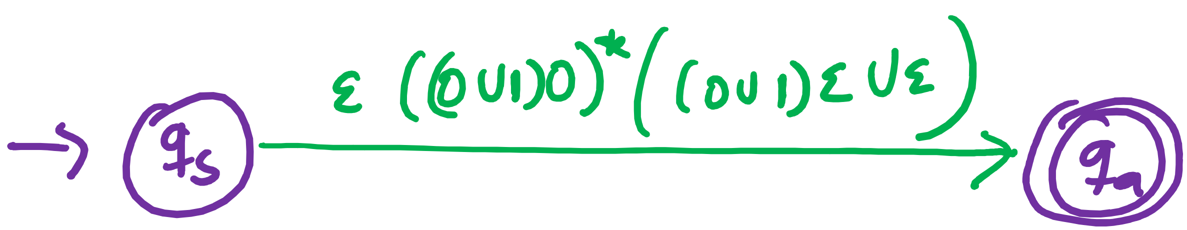


2) Remove q_1



3) Remove q_0

$= ((0\cup1)0)^*(0\cup1\cup\varepsilon)$



For a language L over $\{0, 1\}$, define the operation $\mathbf{split}(L) = \{x\#y \mid x, y \in L\}$. Show that the regular languages are closed under split

For a language L over alphabet Σ, define the operation $\mathbf{drop}(L) = \{xz \mid xyz \in L \textbf{ where } x, z \in \Sigma^*, y \in \Sigma\}$. Show that the regular languages are closed under $\mathbf{drop}$.

Is the following language regular?
$\{0^n1^n \mid 0 \leq n \leq 2026\}$

Is the following language regular? $\{a^n a^n \mid n \geq 0\}$

How many states does a DFA recognizing $\{0^n 1^n \mid 0 \leq n \leq 2026\}$ require?