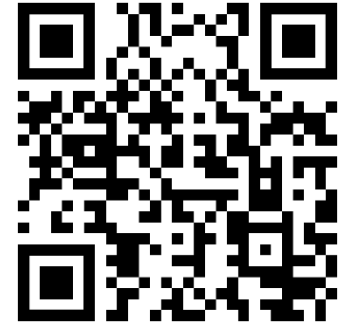


BU CS 332 – Theory of Computation

<https://forms.gle/Xj7E7pXaXdJZEeBc6>



Lecture 9:

- Turing Machines

Reading:

Sipser Ch 3.1-3.3

Mark Bun & Alexander Poremba

February 26, 2026

Turing Machines – Motivation

We've seen finite automata as a restricted model of computation

Finite Automata / Regular Expressions

- Can do simple pattern matching (e.g., substrings), check parity, addition
- Can't perform unbounded counting
- Can't recognize palindromes

$\{0^n 1^n \mid n \geq 0\}$ not regular
 $\{w \mid w^R = w\}$ not regular

Somewhat more powerful (not in this course):

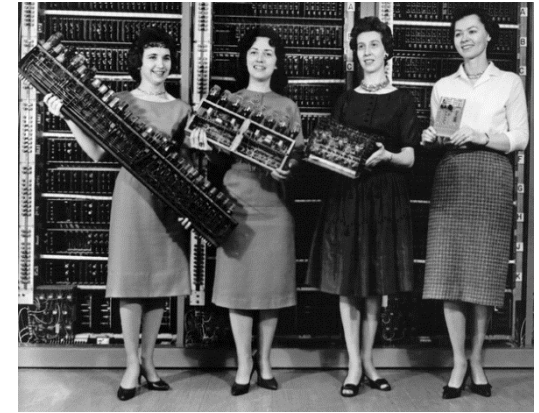
Pushdown Automata / Context-Free Grammars

- Can count and compare, parse math expressions
- Can't recognize $\{a^n b^n c^n \mid n \geq 0\}$

Turing Machines – Motivation

Goal:

Define a model of computation that is



- 1) **General purpose.** Captures all algorithms that can be implemented in any programming language.
- 2) **Mathematically simple.** We can hope to prove that things are not computable in this model.

A Brief History

1900 – Hilbert’s Tenth Problem

Ex: (Diophantine equation)

$$P(x, y, z) = 3xz - x^2y$$

Does there exist $(x, y, z) \in \mathbb{Z}^3$ s.t.

$$P(x, y, z) = 0 ?$$

$(x, y, z) = (1, 3, 1)$ satisfies $P(1, 3, 1) = 0$.

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

Algorithm



David Hilbert 1862-1943

1928 – The *Entscheidungsproblem*



Wilhelm Ackermann 1896-1962

The “Decision Problem”

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

Given: mathematical statement

Output: Is the statement true or false?



David Hilbert 1862-1943

1936 – Solution to the *Entscheidungsproblem*



Alonzo Church 1903-1995

"An unsolvable problem of elementary number theory"

Model of computation: λ -calculus (CS 320)



Alan Turing 1912-1954

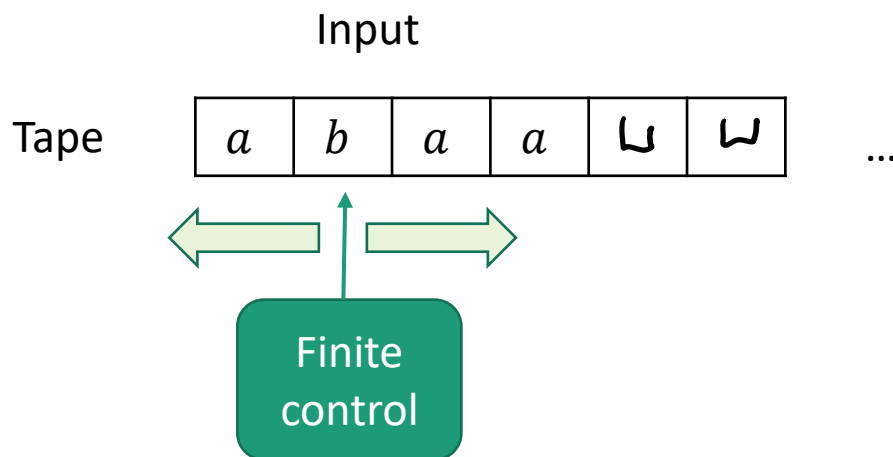
"On computable numbers, with an application to the *Entscheidungsproblem*"

Model of computation: Turing Machine

The Turing Machine Model

The Basic Turing Machine (TM)

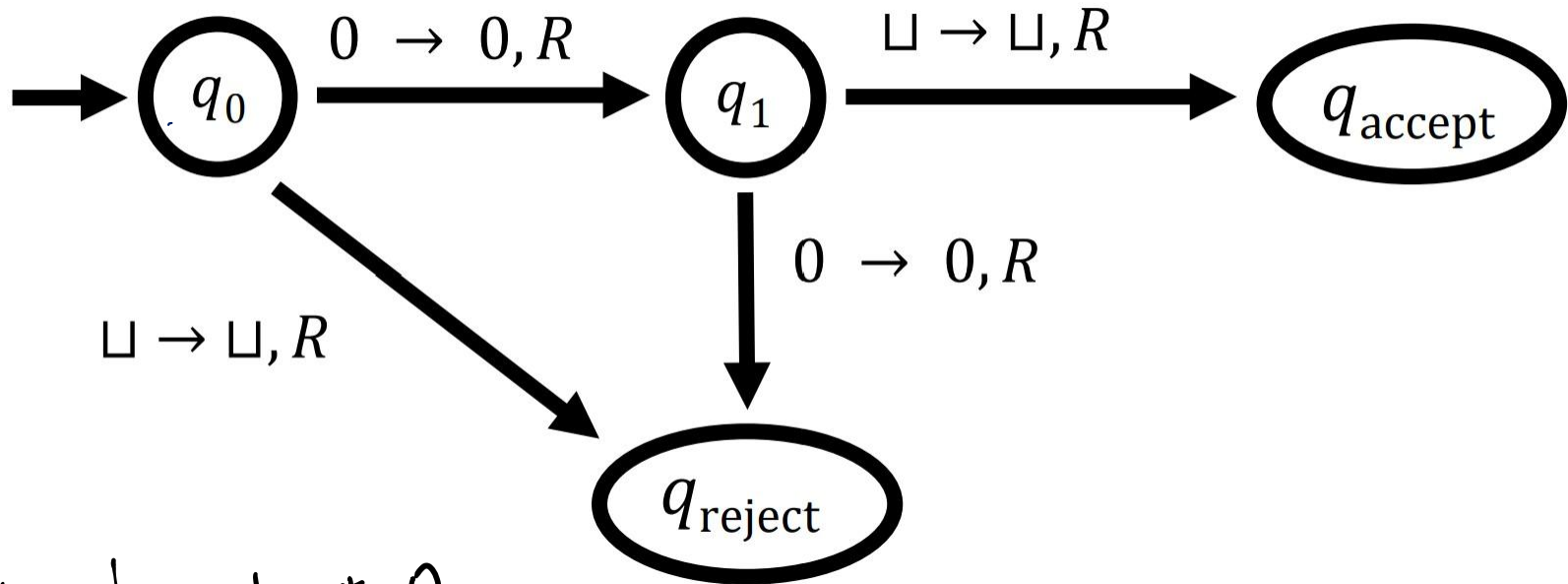
$\Sigma = \{a, b\}$



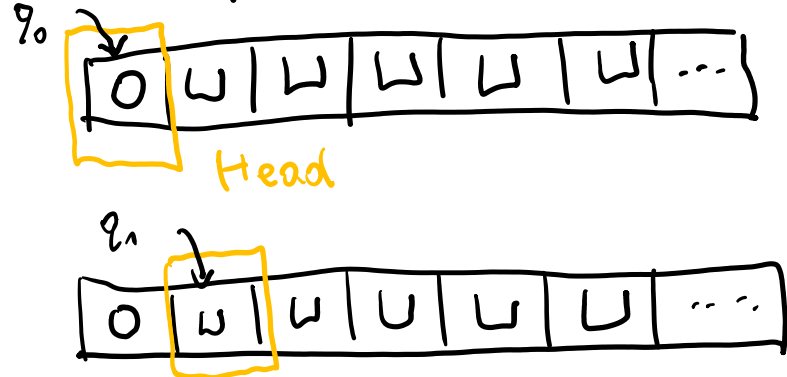
blank
symbol "␣"

- Input is written on an infinitely long tape
- Head can both read and write, and move in both directions
- Computation halts as soon as control reaches "accept" or "reject" state

Example

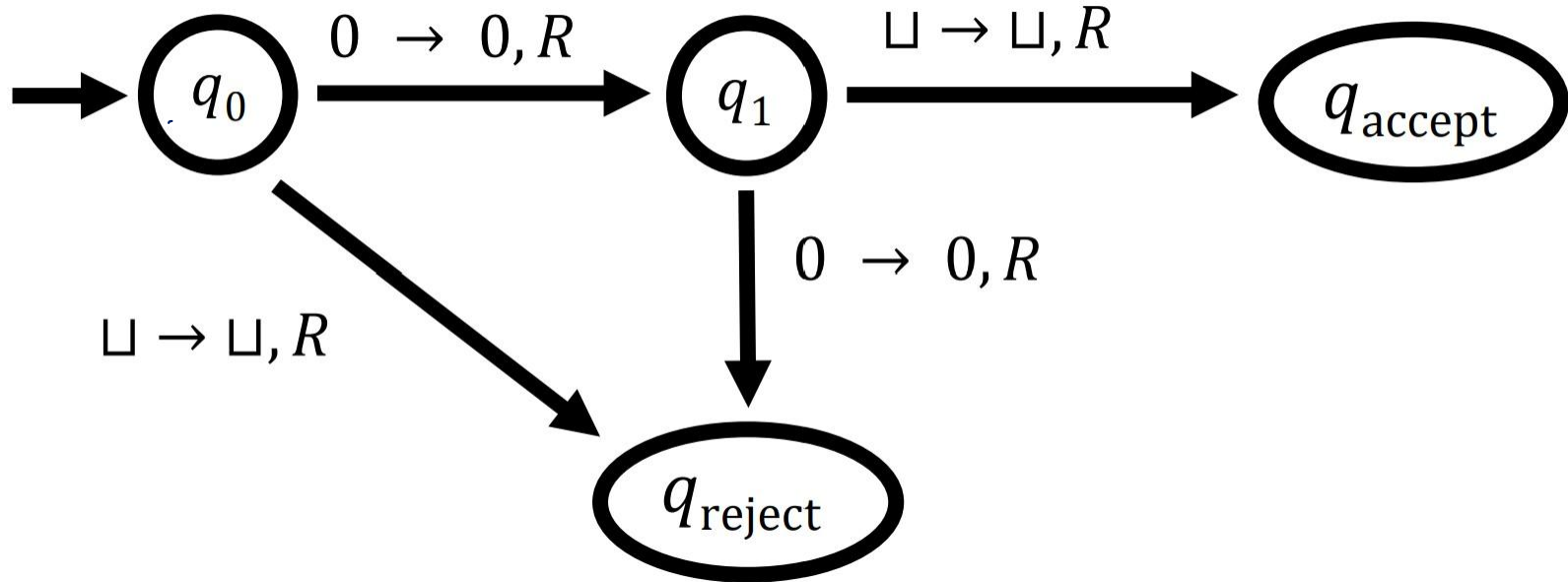


Example: Input 0

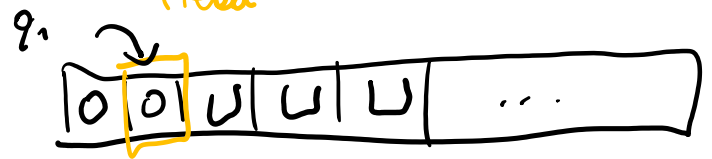
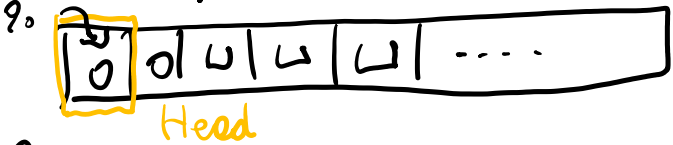


Accepts input "0"
The language recognized by this TM is $L = \{0\}$.

Example

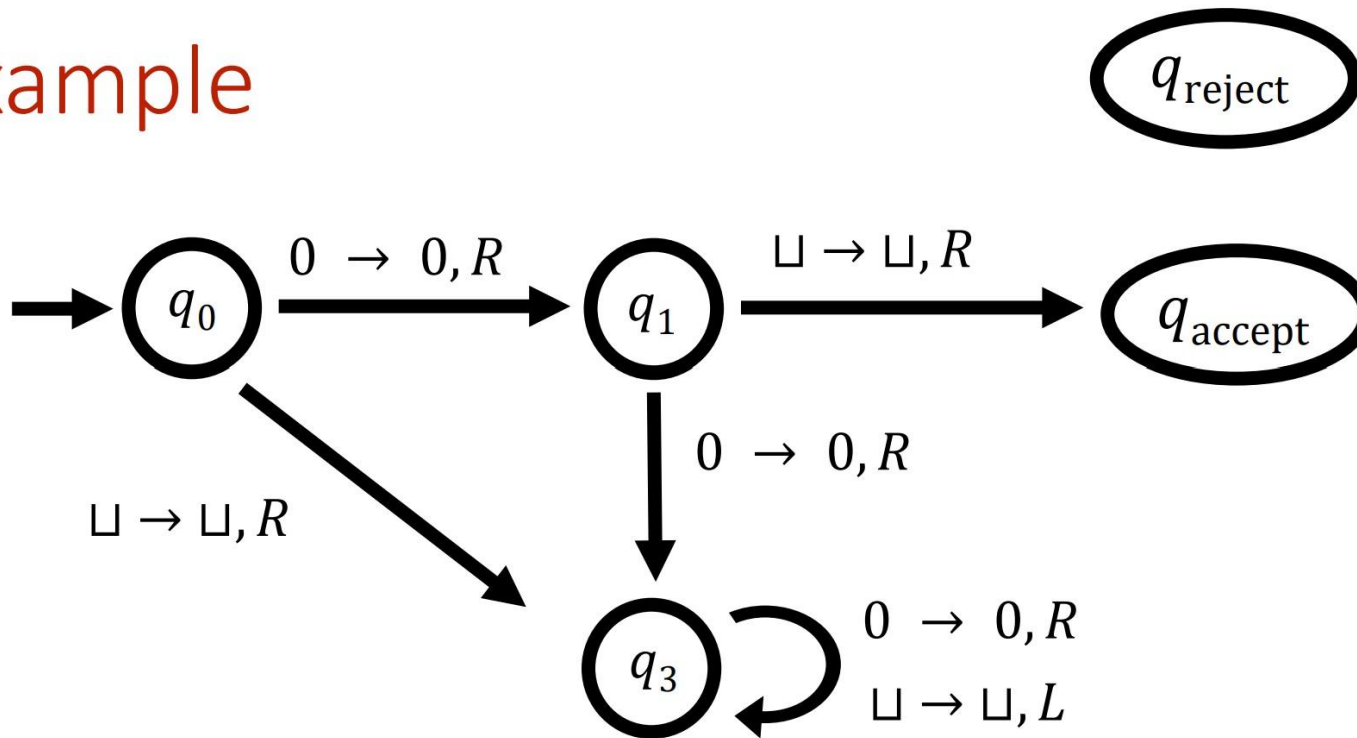


Example: Input 00



TM rejects the input 00.

Example



What does this TM do on input 000?

- a) Halt and accept
- b) Halt and reject
- c) Halt in state q_3
- d) Loop forever without halting



Three Levels of Abstraction

High-Level Description

An algorithm (like CS 330)

*Programming ecology:
Python, Java*

Implementation-Level Description

Describe (in English) the instructions for a TM

- How to move the head
- What to write on the tape

*Assembly
language*

Low-Level Description

State diagram or formal specification

*machine
code*

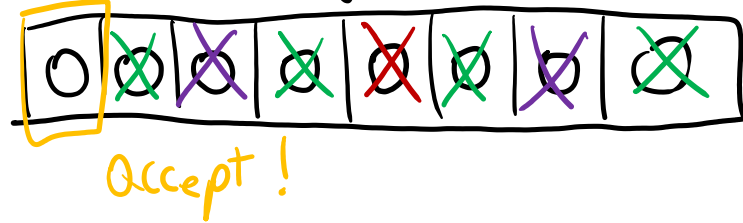
Example

Determine if a string $w \in \{0\}^*$ is in the language

$$A = \{0^{2^n} \mid n \geq 0\}$$

Problem: Given a string of all 0's, is its length a power of 2?

High-Level Description



Repeat the following forever:

- If there is exactly one 0 in w , **accept**
- If there is an odd (> 1) number of 0s in w , **reject**
- Delete half of the 0s in w

Example

Determine if a string $w \in \{0\}^*$ is in the language

$$A = \{0^{2^n} \mid n \geq 0\}$$

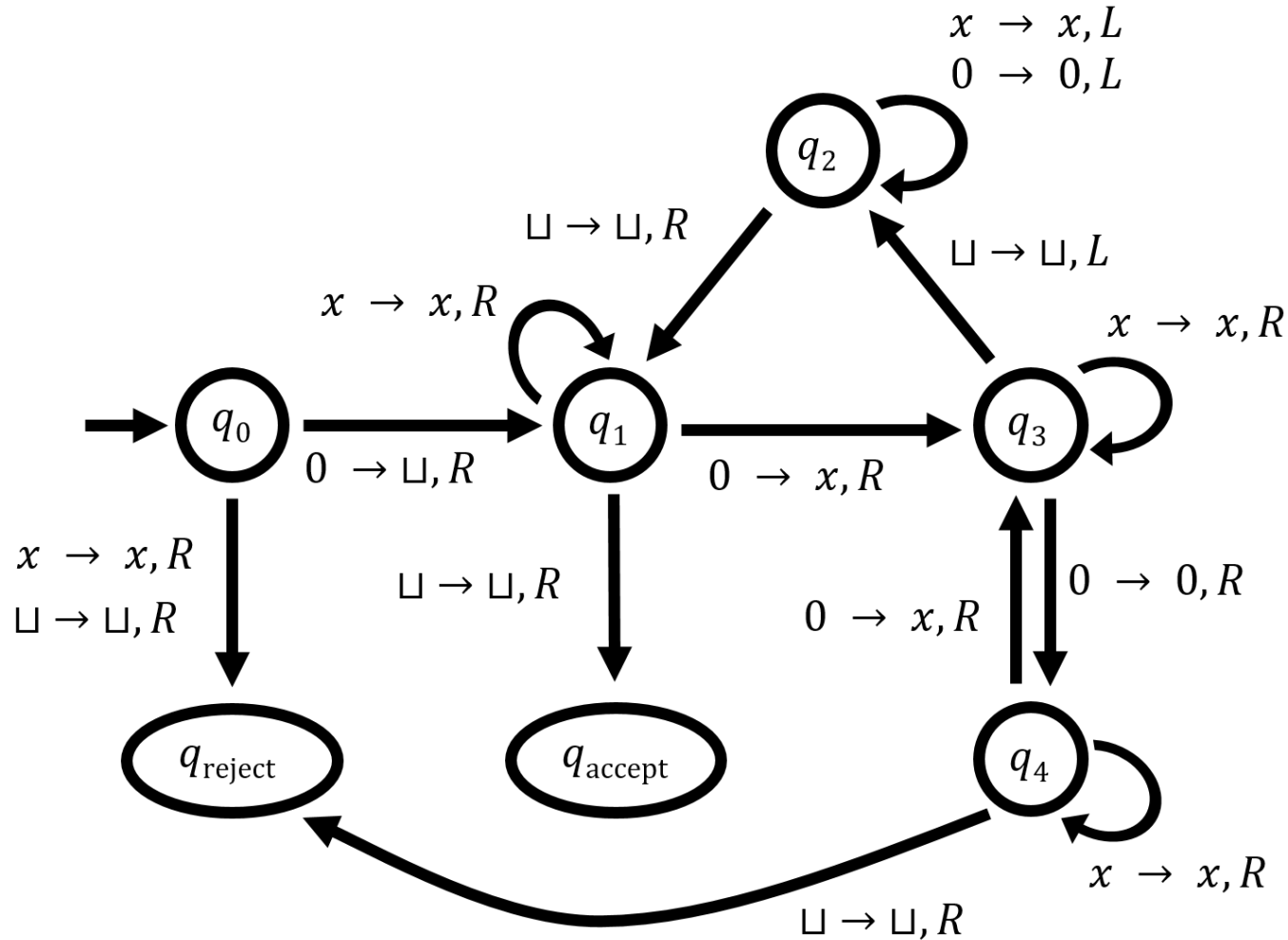
Implementation-Level Description

1. While moving the tape head left-to-right:
 - a) Cross off every other 0
 - b) If there is exactly one 0 when we reach the first blank symbol, **accept**
 - c) If there is an odd (> 1) number of 0s when we reach first blank symbol, **reject**
2. Return the head to the left end of the tape
3. Go back to step 1

Example

Determine if a string $w \in A = \{0^{2^n} \mid n \geq 0\}$

Low-Level Description



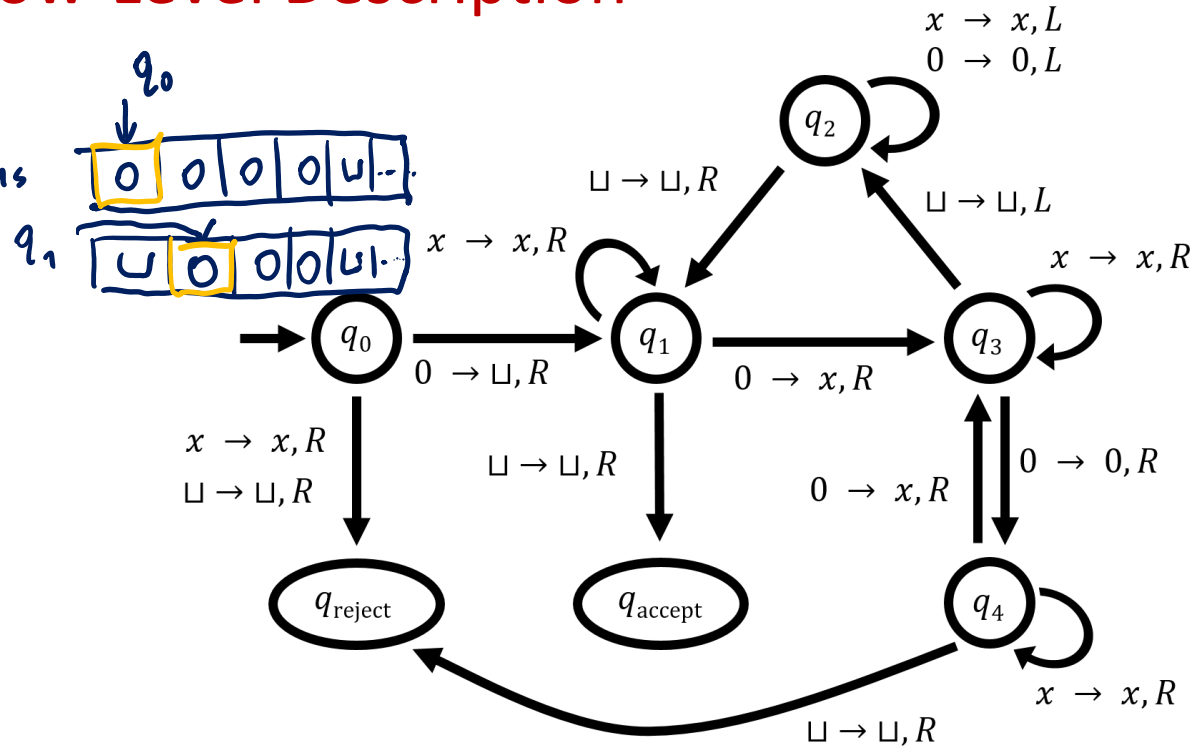
Example

Determine if a string $w \in A = \{0^{2^n} \mid n \geq 0\}$

Low-Level Description

Input: $w = 0000$

q_0 0 0 0 0 \sqcup ... means
 \sqcup q_1 0 0 0 \sqcup ...



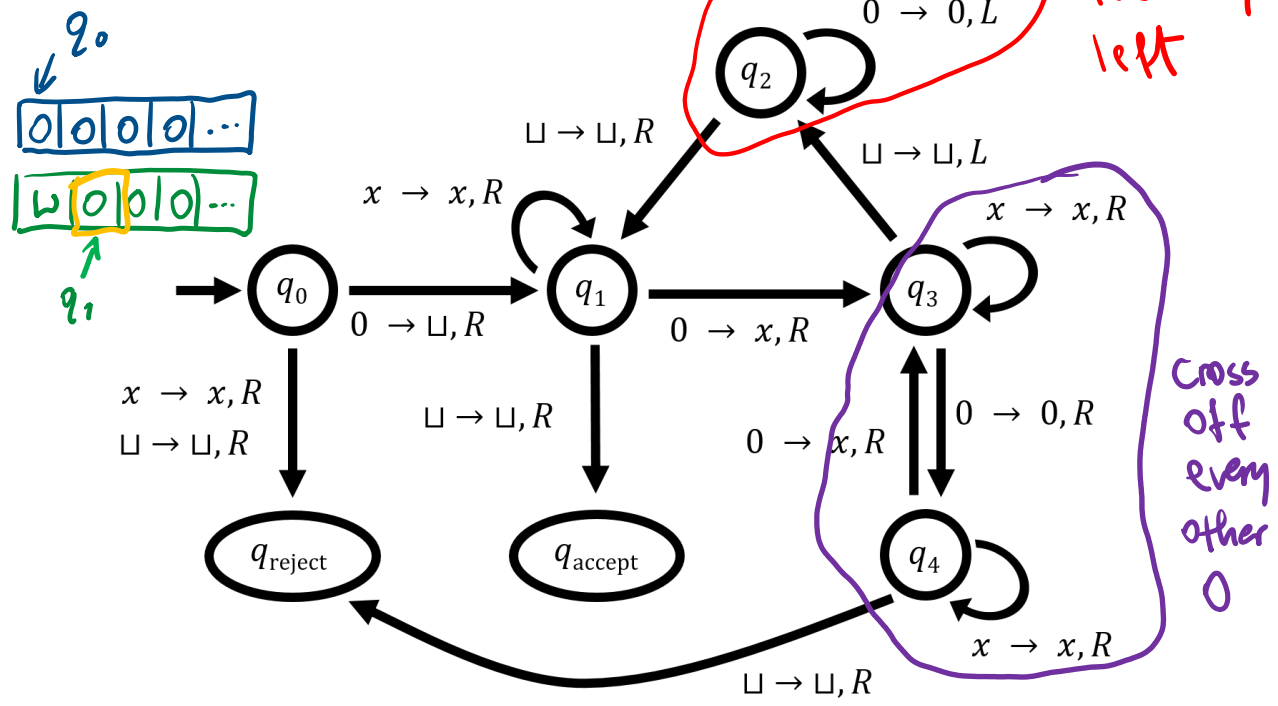
Example

Determine if a string $w \in A = \{0^{2^n} \mid n \geq 0\}$

Low-Level Description

Input: $w = 0000$

q_0 0000 \sqcup ... means
 \sqcup q_1 000 \sqcup ...
 \sqcup x q_3 00 \sqcup ...
 \sqcup x 0 q_4 0 \sqcup ...
 \sqcup x 0 x q_3 \sqcup ...
 \sqcup x 0 q_2 x \sqcup ...
 \sqcup x q_2 0 x \sqcup ...
 \sqcup q_2 x 0 x \sqcup ...
 q_2 \sqcup x 0 x \sqcup ...
 \sqcup q_1 x 0 x \sqcup ...
 \sqcup x q_1 0 x \sqcup ...
 \sqcup x x q_3 x \sqcup ...
 \sqcup x x x q_3 \sqcup ...



\sqcup x x q_2 x \sqcup ...
 \sqcup x q_2 x x \sqcup ...
 \sqcup q_2 x x x \sqcup ...
 q_2 \sqcup x x x \sqcup ...
 \sqcup q_1 x x x \sqcup ...

\sqcup x q_1 x x \sqcup ...
 \sqcup x x q_1 x \sqcup ...
 \sqcup x x x q_1 \sqcup ...
 \sqcup x x x \sqcup q_{accept} \sqcup ...

ACCEPT!

Differences between TMs and Finite Automata

TM halts immediately on reading accept or reject.

TMs can write on the tape.

TMs can move left and right (symbols: L and R)

TMs can enter infinite loops.

TMs are deterministic

Formal Definition of a TM

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q is a finite set of states
- Σ is the input alphabet (does **not** include \sqcup)
- Γ is the tape alphabet (contains \sqcup and Σ)
- δ is the transition function

...more on this later

- $q_0 \in Q$ is the start state
- $q_{\text{accept}} \in Q$ is the accept state
- $q_{\text{reject}} \in Q$ is the reject state ($q_{\text{reject}} \neq q_{\text{accept}}$)

TM Transition Function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

L means “move left” and R means “move right”

$\delta(p, a) = (q, b, R)$ means:

- Replace a with b in current cell
- Transition from state p to state q
- Move tape head right

$\delta(p, a) = (q, b, L)$ means:

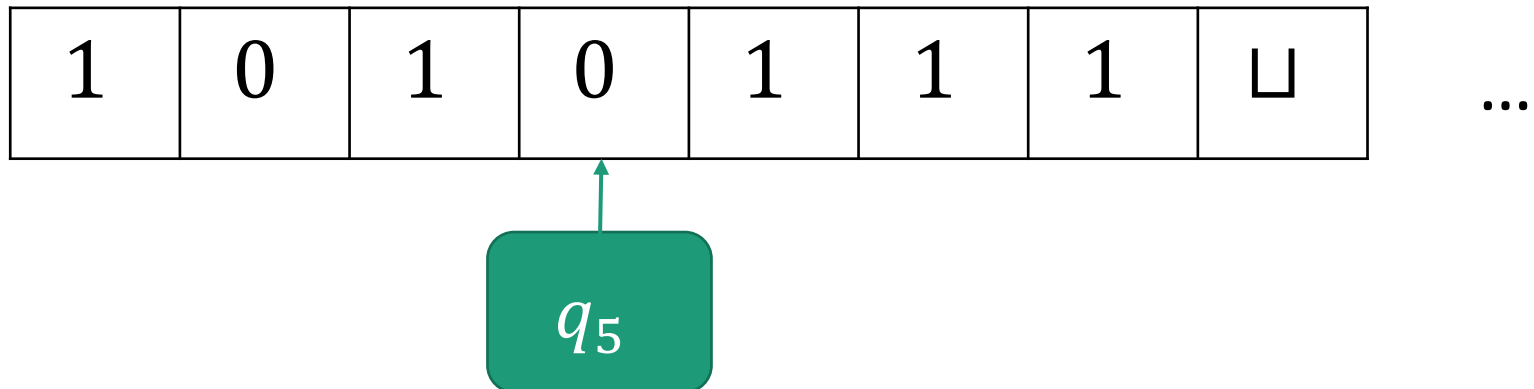
- Replace a with b in current cell
- Transition from state p to state q
- Move tape head left UNLESS we are at left end of tape, in which case don't move

Configuration of a TM: Formally

A **configuration** is a string uqv where $q \in Q$ and $u, v \in \Gamma^*$

- Tape contents = uv (followed by infinitely many blanks \sqcup)
- Current state = q
- Tape head on first symbol of v

Example: $101q_50111$



How a TM Computes



Start configuration: q_0w

In one step of computation:

- If $\delta(q, b) = (q', c, R)$, then $uaqbv$ yields $uacq'v$
- If $\delta(q, b) = (q', c, L)$, then $uaqbv$ yields $uq'acv$
- If we are at the left end of the tape in configuration qbv , what configuration do we reach if $\delta(q, b) = (q', c, L)$?

- a) $cq'v$
- b) $q'cv$
- c) $q' \sqcup cv$
- d) $q'cbv$

How a TM Computes

Start configuration: q_0w

In one step of computation:

- If $\delta(q, b) = (q', c, R)$, then $uaqbv$ yields $uacq'v$
- If $\delta(q, b) = (q', c, L)$, then $uaqbv$ yields $uq'acv$
- If $\delta(q, b) = (q', c, L)$, then qbv yields $q'cv$

Accepting configuration: $q = q_{\text{accept}}$

Rejecting configuration: $q = q_{\text{reject}}$