BU CS 332 – Theory of Computation

https://forms.gle/MBf73tYPrgZNcfYm9

Lecture 13:

- Uncountability
- Undecidability

Reading:

Sipser Ch 4.2, 5.1

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Where we are and where we're going

Church-Turing thesis: TMs capture all algorithms

Consequence: studying the limits of TMs reveals the limits of computation

Last time: Sizes of infinite sets, countability

Today: Uncountable sets

Existential proof that there are undecidable and unrecognizable languages

An explicit undecidable language?

How can we compare sizes of infinite sets?

Definition: Two sets have the same size if there is a bijection between them

A set is countable if either

- it is a finite set, or
- it has the same size as $\mathbb{N}$, the set of natural numbers

How to show that $\mathbb{N} \times \mathbb{N}$ is countable?

f(1)= f(2)= f(4)= f(3)= f(5)=5

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	…
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	…
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	…
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	…
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	
					$\ddots$

How to argue that a set S is countable

- Describe how to "list" the elements of S, usually in stages:

Ex: Stage 1) List all pairs (x, y) such that $x + y = 2$

Stage 2) List all pairs (x, y) such that $x + y = 3$

…

Stage n) List all pairs (x, y) such that $x + y = n + 1$

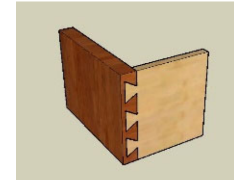
…

- Explain why every element of S appears in the list

Ex: Any $(x, y) \in \mathbb{N} \times \mathbb{N}$ will be listed in stage $x + y - 1$

- Define the bijection $f: \mathbb{N} \to S$ by $f(n) =$ the n'th element in this list (ignoring duplicates if needed)

Another version of the dovetailing trick



Ex: Show that $\mathcal{F} = \{L \subseteq \{0,1\}^* \mid L \text{ is finite}\}$ is countable

$\{\varepsilon, 0, 01\} \in \mathcal{F}$ $\{0^n \mid n \geq 0\} \notin \mathcal{F}$

<u>Proof 1:</u> Encode an arbitrary finite language $L = \{x_1, x_2, \ldots, x_n\}$ as a single string $x_1 \# x_2 \# \ldots \# x_n \in \{0, 1, \#\}^*$

Induces an injective function $f: \mathcal{F} \to \{0, 1, \#\}^*$

Know $\{0, 1, \#\}^*$ is countable, and if $\exists$ an injective function from a set into a countable set, that set is countable.

<u>Proof 2:</u> For a finite language L, let $m(L) =$ length of longest string in L

List elements of $\mathcal{F}$ in stages:

<u>Stage 1:</u> List all languages L s.t. $|L| \leq 1$ and $m(L) \leq 1$ $\emptyset, \{\varepsilon\}, \{0\}, \{1\}$

<u>Stage 2:</u> List all languages L s.t. $|L| \leq 2$ and $m(L) \leq 2$ $\emptyset, \{\varepsilon\}, \{0\}, \{1\}$, $\{00\}, \{01\}, \{10\}, \{11\}$, $\{\varepsilon, 0\}, \{\varepsilon, 1\}, \{\varepsilon, 00\}$, $\{\varepsilon, 01\}, \ldots$

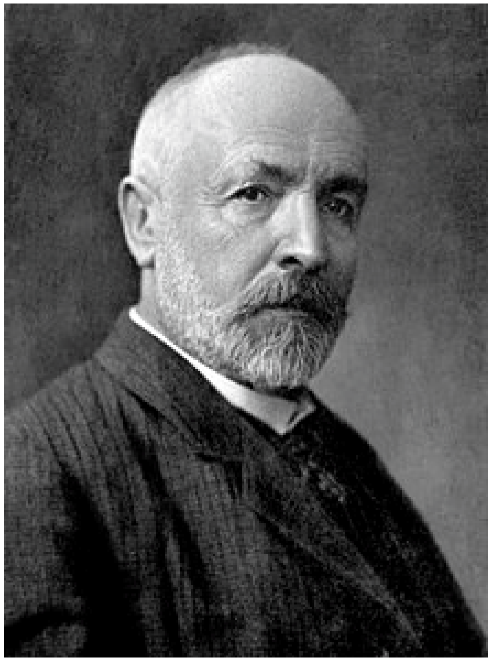
⋮

<u>Stage n:</u> List all L s.t. $|L| \leq n$ and $m(L) \leq n$

Bijection $f: \mathbb{N} \to \mathcal{F}$ defined by $f(i) =$ i'th distinct language listed.

So what *isn't* countable?

Cantor's Diagonalization Method



Georg Cantor 1845-1918

- Invented set theory
- Defined countability, uncountability, cardinal and ordinal numbers, …

Some praise for his work:

"Scientific charlatan…renegade…corruptor of youth"
–L. Kronecker

"Set theory is wrong…utter nonsense…laughable"
–L. Wittgenstein

Uncountability of the reals

Theorem: The real interval $[0,1]$ is uncountable.

Proof: We'll show that there is no surjection $\mathbb{N} \to [0,1]$.

Let $f : \mathbb{N} \to [0,1]$ be an arbitrary function

n	$f(n)$
1	$0\,.\,d_1^1\ d_2^1\ d_3^1\ d_4^1\ d_5^1 \ldots$
2	$0\,.\,d_1^2\ d_2^2\ d_3^2\ d_4^2\ d_5^2 \ldots$
3	$0\,.\,d_1^3\ d_2^3\ d_3^3\ d_4^3\ d_5^3 \ldots$
4	$0\,.\,d_1^4\ d_2^4\ d_3^4\ d_4^4\ d_5^4 \ldots$
5	$0\,.\,d_1^5\ d_2^5\ d_3^5\ d_4^5\ d_5^5 \ldots$
⋮	

Ex:

$f(1) = 0.314159\ldots$

$f(2) = 0.221828\ldots$

$f(3) = 0.8675309\ldots$

⋮

$b = 0.488\ldots$

Construct $b \in [0,1]$ which does not appear as any $f(n)$

– Then f can't be a surjection!

$b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$ (digit n of $f(n)$)

e.g. $b_n = d_n^n + 1 \pmod{10}$

Diagonalization

This process of constructing a counterexample by "contradicting the diagonal" is called diagonalization

Structure of a diagonalization proof

Say you want to show that a set T is uncountable

1) Let $f: \mathbb{N} \to T$ be an arbitrary function
2) "Flip the diagonal" to construct an element $b \in T$ such that $f(n) \neq b$ for every n

Ex: Let $b = 0.b_1 b_2 b_3 \ldots$ where $b_n \neq d_n^n$

(where d_n^n is digit n of $f(n)$)

3) Conclude that f is not a surjection. Since f was arbitrary, there is no surjection from $\mathbb{N} \to T$ so T is not countable

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

$= \{ S \mid S \subseteq X \}$

Proof: Let $f : X \to P(X)$ be arbitrary. We'll show that f is not a surjection

What should we do to show f isn't a surjection?

a) Show that for every $S \in P(X)$, there exists $x \in X$ such that $f(x) = S$

b) Construct a set $S \in P(X)$ (meaning, $S \subseteq X$) that cannot be the output $f(x)$ for any $x \in X$

c) Construct a set $S \in P(X)$ and two distinct $x, x' \in X$ such that $f(x) = f(x') = S$

Diagonalization argument

Let $f: X \rightarrow P(X)$ be an arbitrary function

Elements of X

x					
x_1					
x_2					
x_3					
x_4					
$\vdots$					

Diagonalization argument

Let $f: X \to P(X)$ be an arbitrary function

Is $x_3 \in f(x_2)$?

x	$x_1 \in f(x)$?	$x_2 \in f(x)$?	$x_3 \in f(x)$?	$x_4 \in f(x)$?	…
x_1	~~Y~~ N	N	Y	Y	
x_2	N	~~N~~ Y	Y	Y	
x_3	Y	Y	~~Y~~ N	N	
x_4	N	N	Y	~~N~~ Y	
$\vdots$					$\ddots$

$S = \{x_2, x_4, \ldots\}$

Define S by flipping the diagonal:

Put $\quad x_i \in S \quad \Longleftrightarrow \quad x_i \notin f(x_i)$

Example

Let $X = \{1, 2, 3\}$, $P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Ex. $f(1) = \{1, 2\}$, $f(2) = \emptyset$, $f(3) = \{2\}$

x	$1 \in f(x)$?	$2 \in f(x)$?	$3 \in f(x)$?
1	~~Y~~ N	Y	N
2	N	~~N~~ Y	N
3	N	Y	~~N~~ Y

- $S \neq f(1)$ because $1 \in f(1) = \{1, 2\}$, $1 \notin S$
- $S \neq f(2)$ because $2 \notin f(2) = \emptyset$, $2 \in S$
- $S \neq f(3)$ because $3 \notin f(3) = \{2\}$, $3 \in S$

Construct $S = \{2, 3\}$

A general theorem about set sizes

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

This is complete solution ... no need to draw a table

Proof: Let $f: X \to P(X)$ be an arbitrary function.

Define

$$S = \{x \in X \mid x \notin f(x)\} \quad \in P(X)$$

Suppose, for contradiction, that $S = f(y)$ for some $y \in X$

Then $y \in S$ if and only if $y \notin S$ $\lightning$

Hence $S \in P(X)$ cannot be the output $f(x)$ for any $x \in X$, so f is not a surjection.

Undecidable Languages

Undecidability / Unrecognizability

Definition: A language L is **undecidable** if there is no TM deciding L

Definition: A language L is **unrecognizable** if there is no TM recognizing L

An existential proof

i.e. $L \subseteq \{0,1\}^*$

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0, 1\}^*$

Set of all languages over $\{0, 1\}$:

a) $\{0, 1\}$

b) $\{0, 1\}^*$

c) $P(\{0, 1\}^*)$: The set of all subsets of $\{0, 1\}^*$

d) $P(P(\{0, 1\}^*))$: The set of all subsets of the set of all subsets of $\{0, 1\}^*$

An existential proof

Theorem: There exists an undecidable language over $\{0, 1\}$

Proof:

Set of all encodings of TM deciders: $X \subseteq \{0, 1\}^*$

Set of all languages over $\{0, 1\}$: $P(\{0, 1\}^*)$

There are more languages than there are TM deciders!

$\Rightarrow$ There must be an undecidable language

An existential proof

Theorem: There exists an unrecognizable language over $\{0,1\}$

Proof:

Set of all encodings of TMs: $X \subseteq \{0,1\}^*$

Set of all languages over $\{0,1\}$: $P(\{0,1\}^*)$

There are more languages than there are TM recognizers!

$\Rightarrow$ There must be an unrecognizable language

“Almost all” languages are undecidable

But how do we actually find one?

An Explicit Undecidable Language

Our power set size proof

Theorem: Let X be any set. Then the power set $P(X)$ does **not** have the same size as X.

1) Let $f: X \to P(X)$ be an arbitrary function

2) "Flip the diagonal" to construct a set $S \in P(X)$ such that $f(x) \neq S$ for every $x \in X$ $\quad S = \{x \mid x \notin f(x)\}$

3) Conclude that f is not a surjection

Specializing the proof

Theorem: Let X be the set of all TM deciders. Then there exists an undecidable language in $P(\{0,1\}^*)$

1) Consider the function $L: X \to P(\{0,1\}^*)$

   L(M) = language decided by TM M

2) "Flip the diagonal" to construct a language $UD \in P(\{0,1\}^*)$ such that $L(M) \neq UD$ for every $M \in X$

3) Conclude that L is not a surjection

An explicit undecidable language

TM M					
M_1					
M_2					
M_3					
M_4					
$\vdots$					

Why is it possible to enumerate all TMs like this?

a) The set of all TMs is finite

b) The set of all TMs is countably infinite

c) The set of all TMs is uncountable

An explicit undecidable language

$M(\langle R \rangle)$ means: run TM M on input $\langle R \rangle$

$\begin{cases} Y & \text{if } M_2 \text{ accepts input } \langle M_3 \rangle \\ N & \text{if } M_2 \text{ rejects input } \langle M_3 \rangle \end{cases}$

TM M	$M(\langle M_1 \rangle)$?	$M(\langle M_2 \rangle)$?	$M(\langle M_3 \rangle)$?	$M(\langle M_4 \rangle)$?		$D(\langle D \rangle)$?
M_1	~~Y~~ N	N	Y	Y	…	
M_2	N	~~N~~ Y	Y	Y		
M_3	Y	Y	~~Y~~ N	N		
M_4	N	N	Y	~~N~~ Y		
⋮					⋱	
D						~~Y~~ N | ~~N~~ Y

$UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle\}$

Claim: UD is undecidable

Let D be an arbitrary TM decider

Case 1: D accepts $\langle D \rangle \Rightarrow \langle D \rangle \notin UD$ by defn of UD. D made a mistake on input $\langle D \rangle$!

Case 2: D rejects $\langle D \rangle \Rightarrow \langle D \rangle \in UD$ by defn of UD. D made a mistake on input $\langle D \rangle$

D can't decide UD because it's wrong on input $\langle D \rangle$.

An explicit undecidable language

Theorem: $UD = \{\langle M \rangle \mid M \text{ is a TM that does not accept on input } \langle M \rangle\}$ is undecidable

Proof: Suppose for contradiction that TM D decides UD