CS 535: Complexity Theory, Fall 2020

Homework 5

Due: 8:00PM, Friday, October 16, 2020.

Reminder. Homework must be typeset with LaTeX preferred. Make sure you understand the course collaboration and honesty policy before beginning this assignment. Collaboration is permitted, but you must write the solutions *by yourself without assistance*. You must also identify your collaborators. Assignments missing a collaboration statement will not be accepted. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Problem 1 (Alternation)**.**

(a) Let $k \in \mathbb{N}$. Prove that a language $L \in \mathbf{\Sigma_2 TIME}(n^k)$ if and only if there exists a constant c and a (deterministic) TM $M(x, u, v)$ running in $O(|x|^k)$ steps such that

$$x \in L \iff \exists u \in \{0,1\}^{c|x|^k} \forall v \in \{0,1\}^{c|x|^k} M(x, u, v) = 1.$$

(3 points)

(b) Prove that $\mathbf{\Sigma_2^p} = \cup_{k=1}^{\infty} \mathbf{\Sigma_2 TIME}(n^k)$. (2 points)

(c) Let $s(n) \geq n$ be space constructible. Prove that $\mathbf{NSPACE}(s(n)) \subseteq \mathbf{ATIME}((s(n))^2)$. Hint: Recall the proof of Savitch's Theorem. (6 points)

Problem 2 (Polynomial Hierarchy)**.**

(a) Define the language

$$\exists\mathsf{USAT} = \{\varphi \text{ a Boolean formula} \mid \exists x \in \{0,1\}^n \ \exists! y \in \{0,1\}^m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\}.$$

Here, the notation "$\exists!$" means "there exists exactly one" (satisfying y). For example, $\varphi(x, y_1, y_2) = (x \wedge y_1 \wedge \overline{y_2}) \vee (\bar{x} \wedge y_1) \in \exists\mathsf{USAT}$ because setting $x = 1$ makes $y_1 = 1$ and $y_2 = 0$ the unique satisfying assignment to the formula. Show that $\exists\mathsf{USAT}$ is $\mathbf{\Sigma_2^p}$-complete. (6 points)

(b) Suppose that one day, science shows that $\mathbf{\Sigma_6^p} \subseteq \mathbf{\Pi_4^p}$. Show that the polynomial hierarchy collapses, to the lowest level that you can. (3 points)

Problem 3 (* Bonus * Our Pal **AL**)**.** Let $\mathbf{AL} = \mathbf{ASPACE}(\log n)$ be the class of languages decidable in logarithmic space by an alternating TM. Prove that $\mathbf{P} = \mathbf{AL}$.