Problem 1 (Perfect Interactive Proofs). For parameters $c, s \geq 0$, define the class $\text{MA}_{c,s}$ to consist of Merlin-Arthur interactive proofs with completeness probability $c$ and soundness probability $s$. That is, a language $L \in \text{MA}_{c,s}$ if there exists a probabilistic poly-time verifier $V$ and a polynomial $m(n)$ such that

- $x \in L \iff \exists u \in \{0, 1\}^{m(n)} \Pr[V(x, u) = 1] \geq c$
- $x /\in L \iff \forall u \in \{0, 1\}^{m(n)} \Pr[V(x, u) = 1] \leq s$

Recall that in class we defined $\text{MA} = \text{MA}_{2/3,1/3}$.

(a) Prove that $\text{MA}_{1,1/3} = \text{MA}$. That is, we may assume Merlin-Arthur proofs have perfect completeness probability. Hint: Modify the proof of the Sipser-Gács Theorem (Theorem 7.15). (6 points)

(b) Prove that $\text{MA}_{2/3,0} = \text{NP}$. That is, Merlin-Arthur proofs with perfect soundness are no more powerful than deterministic proofs. (4 points)

(c) *Bonus* Prove the same relationships for general interactive proofs. That is, show that $\text{IP}_{1,1/3} = \text{IP}$ and $\text{IP}_{2/3,0} = \text{NP}$.

Problem 2 (Counting k-Colorings). Let $G = ([n], E)$ be a graph on $n$ vertices. A $k$-coloring of $G$ is a vector of colors $(c_1, \ldots, c_n) \in [k]^n$ such that for every edge $(i, j) \in E$, we have $c_i \neq c_j$.

(a) Show that there is a degree-$kn$ integer polynomial $p$ such that the number of $k$-colorings of a graph $G$ is given by

$$\sum_{c_1=1}^{k} \sum_{c_2=1}^{k} \cdots \sum_{c_n=1}^{k} p(c_1, \ldots, c_k).$$


(b) Modify the sumcheck protocol to show that for every constant $k$, the language $\#k\text{COL}_D = \{\langle G, t \rangle \mid G \text{ has exactly } t \text{ } k\text{-colorings} \} \in \text{IP}$. (6 points)