CS 535: Complexity Theory, Fall 2023

Homework 2

Due: 11:59PM, Tuesday, September 12, 2023.

Reminder. Homework must be typeset with LaTeX preferred. Make sure you understand the course collaboration and honesty policy before beginning this assignment. Collaboration is permitted, but you must write the solutions *by yourself without assistance*. You must also identify your collaborators. Assignments missing a collaboration statement will not be accepted. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Problem 1 (coNP). Recall that the complexity class **coNP** consists of languages L such that $\overline{L} \in \mathbf{NP}$.

(a) Show that a language L is **NP**-complete if and only if $\overline{L}$ is **coNP**-complete. Recall that completeness for both classes is defined with respect to polynomial-time (Karp) reductions. (3 points)

(b) Recall that the set difference operation on two languages A, B is defined by $A \setminus B = \{x \mid x \in A, x \notin B\}$. Is **NP** closed under the set difference operation? That is, is it the case that for every $A, B \in \mathbf{NP}$, we have $A \setminus B \in \mathbf{NP}$? Justify your answer. (3 points)

(c) Define the language

$$\mathsf{FACTOR} = \{\langle m, k\rangle \mid m \text{ has a prime factor } p \text{ such that } p \leq k\}.$$

Here, the numbers m and k are written in binary. Show that if FACTOR is **NP**-complete, then $\mathbf{NP} = \mathbf{coNP}$.

You can use without proof that there is a polynomial time algorithm for testing whether a number p (written in binary) is prime. (6 points)

(d) Find the first error in the following "proof" that $\mathbf{NP} = \mathbf{coNP}$, and explain why it is an error: Let M be a nondeterministic polynomial-time algorithm deciding

$$\mathsf{FSAT} = \{\varphi \text{ a Boolean formula } \mid \exists x\ \varphi(x) = 1\}.$$

We design a nondeterministic polynomial-time algorithm deciding

$$\mathsf{TAUT} = \{\varphi \text{ a Boolean formula } \mid \forall x\ \varphi(x) = 1\}$$

as follows. On input φ (an instance of TAUT), evaluate $b = M(\neg\varphi)$. If $b = 0$, output 1, and if $b = 1$, output 0. This runs in nondeterministic polynomial-time as long as M does, and $\varphi \in \mathsf{FSAT}$ iff $\neg\varphi \notin \mathsf{TAUT}$, so it decides TAUT. Therefore, $\mathsf{TAUT} \in \mathbf{NP}$. Since TAUT is **coNP**-complete, it follows that $\mathbf{coNP} \subseteq \mathbf{NP}$. A similar argument shows that $\mathbf{NP} \subseteq \mathbf{coNP}$, hence $\mathbf{NP} = \mathbf{coNP}$. (2 points)

Problem 2 (Class Assignments)**.** As the principal of an elementary school, one of your jobs is to determine class assignments for the n students entering 2nd grade. The first grade teachers generated a list of "disruptive friend groups," which are subsets $G_1, G_2, \ldots, G_m \subseteq [n]$ such that the presence of any complete friend group G_i in a classroom will be...problematic.

Is it possible to partition the n students into two classrooms such that every disruptive friend group is split between the two classes? That is, do there exist disjoint sets S_1, S_2 such that $S_1 \cup S_2 = [n]$ and for every friend group G_i, we have $G_i \not\subseteq S_1$ and $G_i \not\subseteq S_2$?

Define the language PC (short for "peaceful classrooms") such that an instance $\langle n, G_1, \ldots, G_m \rangle \in$ PC if and only if there exists an appropriate partition of the n students described above. Show that the language PC is **NP**-complete via a poly-time reduction from 3SAT. (8 points)