Problem 1 (Logspace Computability). Let \( f : \{0,1\}^* \rightarrow \{0,1\}^* \). Prove that the following two statements are equivalent:

1. The function \( f \) is logspace computable in the sense that we defined in class. That is, there is a TM with a read-only input tape and write-only/write-once output tape that computes \( f \) using only \( O(\log n) \) cells of its (non-input, non-output) work tapes.

2. The function \( f \) is implicitly logspace computable in the sense of Arora-Barak. That is, \( f \) is polynomially-bounded (i.e., there exists a polynomial \( p \) such that \( |f(x)| \leq p(|x|) \) for every \( x \)) and the languages \( \{ \langle x, i \rangle | i \leq |f(x)| \} \) and \( \{ \langle x, i \rangle | (f(x))_i = 1 \} \) are both in \( L \).

(4 points)

Problem 2 (NL-Completeness).

Recall that a directed graph is a DAG (directed acyclic graph) if it does not contain a directed cycle. Show that the language

\[
\text{DAGPATH} = \{ \langle G, s, t \rangle | G \text{ is a DAG and has a path from } s \text{ to } t \}
\]

is NL-complete. (7 points)

Problem 3 (Boolean and Polynomial Hierarchies).

(a) Define the Boolean hierarchy inductively as follows:

\[
\begin{align*}
\text{BH}_1 &= \text{NP} \\
\text{BH}_{2i} &= \{ A \cap B | A \in \text{BH}_{2i-1}, B \in \text{coNP} \} \\
\text{BH}_{2i+1} &= \{ A \cup B | A \in \text{BH}_{2i}, B \in \text{NP} \} \\
\text{BH} &= \bigcup_{i=1}^{\infty} \text{BH}_i.
\end{align*}
\]

That is, the Boolean hierarchy consists of languages that can be written as a finite combination of unions, intersections, and complements of languages in \( \text{NP} \). Show that \( \text{BH} \subseteq P^{\text{NP}} \subseteq \Sigma_2^p \cap \Pi_2^p \). (3 points)
(b) *Individual Review* (No collaboration allowed for this part only)
Define the language $\text{SATAUT} = \{ \langle \varphi, \psi \rangle \mid \varphi \in \text{SAT}, \psi \in \text{TAUT} \}$. Show that $\text{SATAUT}$ is $\text{BH}_2$-complete under poly-time (Karp) reductions. (4 points)

(c) Show that if the polynomial hierarchy does not collapse, then $\text{P} \subsetneq \text{PH} \subsetneq \text{PSPACE}$. (2 points)

Hint: There’s a one-sentence solution to this given what’s shown in Arora-Barak and in class.

(d) (Bonus) Show that if the Boolean hierarchy collapses, then the polynomial hierarchy collapses.