CS 535: Complexity Theory, Fall 2023

Homework 5

Due: 11:59PM, Tuesday, October 17, 2023.

Reminder. Homework must be typeset with LaTeX preferred. Make sure you understand the course collaboration and honesty policy before beginning this assignment. Collaboration is permitted, but you must write the solutions *by yourself without assistance*. You must also identify your collaborators. Assignments missing a collaboration statement will not be accepted. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Problem 1 (Logspace Computability)**.** Let $f : \{0,1\}^* \to \{0,1\}^*$. Prove that the following two statements are equivalent:

1. The function f is logspace computable in the sense that we defined in class. That is, there is a TM with a read-only input tape and write-only/write-once output tape that computes f using only $O(\log n)$ cells of its (non-input, non-output) work tapes.
2. The function f is implicitly logspace computable in the sense of Arora-Barak. That is, f is polynomially-bounded (i.e., there exists a polynomial p such that $|f(x)| \leq p(|x|)$ for every x) and the languages $\{\langle x, i\rangle \mid i \leq |f(x)|\}$ and $\{\langle x, i\rangle \mid (f(x))_i = 1\}$ are both in $\mathbf{L}$.

(4 points)

Problem 2 (**NL**-Completeness)**.**

Recall that a directed graph is a DAG (directed acyclic graph) if it does not contain a directed cycle. Show that the language

$$\mathsf{DAGPATH} = \{\langle G, s, t\rangle \mid G \text{ is a DAG and has a path from } s \text{ to } t\}$$

is **NL**-complete. (7 points)

Problem 3 (Boolean and Polynomial Hierarchies)**.**

(a) Define the Boolean hierarchy inductively as follows:

$$\begin{aligned}
\mathbf{BH}_1 &= \mathbf{NP} \\
\mathbf{BH}_{2i} &= \{A \cap B \mid A \in \mathbf{BH}_{2i-1}, B \in \mathbf{coNP}\} \\
\mathbf{BH}_{2i+1} &= \{A \cup B \mid A \in \mathbf{BH}_{2i}, B \in \mathbf{NP}\} \\
\mathbf{BH} &= \bigcup_{i=1}^{\infty} \mathbf{BH}_i.
\end{aligned}$$

That is, the Boolean hierarchy consists of languages that can be written as a finite combination of unions, intersections, and complements of languages in **NP**. Show that $\mathbf{BH} \subseteq \mathbf{P}^{\mathbf{NP}} \subseteq \mathbf{\Sigma_2^p} \cap \mathbf{\Pi_2^p}$. (3 points)

(b) ***Individual Review* (No collaboration allowed for this part only)**
Define the language $\mathsf{SATTAUT} = \{\langle \varphi, \psi \rangle \mid \varphi \in \mathsf{SAT}, \psi \in \mathsf{TAUT}\}$. Show that $\mathsf{SATTAUT}$ is $\mathbf{BH}_2$-complete under poly-time (Karp) reductions. (4 points)

(c) Show that if the polynomial hierarchy does not collapse, then $\mathbf{P} \subsetneq \mathbf{PH} \subsetneq \mathbf{PSPACE}$. (2 points)

Hint: There's a one-sentence solution to this given what's shown in Arora-Barak and in class.

(d) **(Bonus)** Show that if the Boolean hierarchy collapses, then the polynomial hierarchy collapses.