CAS CS 535: Complexity Theory

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Lecture Notes 24:

More Hardness of Approximation, Proof of PCP Mini

Reading.

- Arora-Barak § 11.4-11.5

Last time: PCP Theorem, Hardness of Approximation

1 Hardness of Approximation

Theorem 1 (PCP Theorem)**.** *Every language L in $\mathbf{NP}$ has a PCP verifier using $O(\log n)$ random coin tosses and making $q = O(1)$ queries to its certificate.*

Last time we showed that the PCP Theorem is equivalent to the following statement: There exists a constant q such that $\mathsf{Gap}_{1/2}\mathsf{MAX}q\mathsf{CSP}$ is $\mathbf{NP}$-hard.

Using the idea behind the standard $\mathbf{NP}$-hardness reduction from SAT to $\mathsf{3SAT}$, this in turn is equivalent to the statement: There exists a constant $\rho < 1$ such that $\mathsf{Gap}_\rho\mathsf{MAX3SAT}$ is $\mathbf{NP}$-hard.

Just as how we used the $\mathbf{NP}$-hardness of the exact version of $\mathsf{3SAT}$ to prove a host of other $\mathbf{NP}$-hardness results, so too can we use the hardness of the approximation version. For instance:

Theorem 2. *For every constant $\rho \in (0, 1)$, the problem $\mathsf{MAX} - \mathsf{INDSET}$ is $\mathbf{NP}$-hard to approximate within a factor of ρ.*

The quantifier "for every" ρ here is interesting, because the approximation problem becomes easier as $\rho \to 0$.

Proof. First, we'll use the standard $\mathbf{NP}$-hardness reduction from $\mathsf{3SAT}$ to INDSET to show this is true for some $\rho < 1$. Then we'll "amplify" the approximation gap to show that the statement is true for every ρ.

Part 1: Let $\rho < 1$ be such that $\mathsf{Gap}_\rho\mathsf{MAX3SAT}$ is $\mathbf{NP}$-hard. Let f be the standard $\mathbf{NP}$-hardness reduction from $\mathsf{3SAT}$ to INDSET. This reduction has the property that, for every 3CNF formula φ, we have

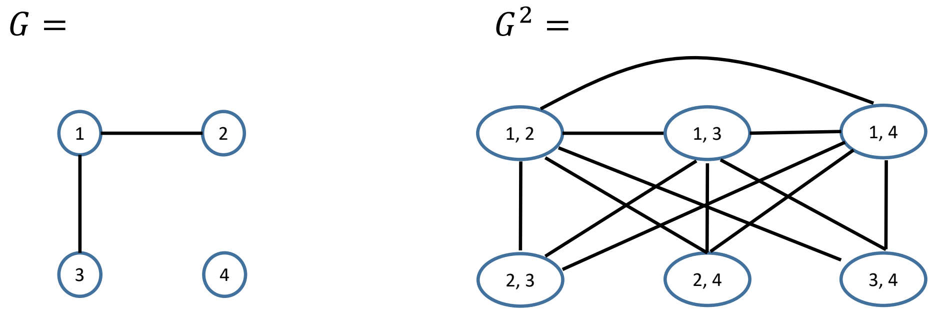
$$\mathrm{val}(\varphi) \geq k \iff f(\varphi) \text{ has an independent set of size } \geq k.$$

Thus, if φ is a formula with m clauses, we have

$$\begin{aligned} \mathrm{val}(\varphi) = m &\implies \mathrm{IS}(f(\varphi)) = m \\ \mathrm{val}(\varphi) < \rho m &\implies \mathrm{IS}(f(\varphi)) < \rho m \end{aligned}$$

where $\mathrm{IS}(G)$ is the size of the largest independent set in G. So it is $\mathbf{NP}$-hard to approximate IS to within a factor of ρ.

Part 2: Now we reduce the problem of ρ-approximating the largest independent set in a graph G to that of ρ^k-approximating the largest independent set in a new graph G^k. The new graph G^k is constructed (in poly-time) from G as follows:
Vertices of G^k: All subsets of k vertices of G
Edges of G^k: $S \sim T$ iff $S \cup T$ is not independent in G.



Let T be a maximum-size independent set in G. Then one can check that the largest independent set in G^k is:

$$\{S \subset T \mid |S| = k\}.$$

Thus we have

$$\mathrm{IS}(G^k) = \binom{\mathrm{IS}(G)}{k} \approx (\mathrm{IS}(G))^k.$$

Thus, we have

$$\begin{aligned} \mathrm{IS}(G) = m &\implies \mathrm{IS}(G^k) \approx m^k \\ \mathrm{IS}(G) < \rho m &\implies \mathrm{IS}(G^k) \lesssim (\rho m)^k = \rho^k m^k. \end{aligned}$$

We can make the gap ρ^k smaller than any constant by taking k to be a sufficiently large constant. $\square$

2 PCP Mini

"The" PCP Theorem says $\mathbf{NP} = \mathbf{PCP}(r(n) = \log n, q(n) = 1)$, where $r(n)$ represents the number of verifier coin tosses, and $q(n)$ is the number of queries. Recall that WLOG, we can always take the length of the proof to be $2^{q(n)} \cdot r(n) = \mathrm{poly}(n)$ in this statement.

Today, we'll prove the following "mini" (or maybe "mega" depending on how you look at it) version of the PCP Theorem:

Theorem 3 (PCP Mini)**.** $\mathbf{NP} \subseteq \mathbf{PCP}(\mathrm{poly}(n), 1)$.

That is, every language in $\mathbf{NP}$ has a PCP system with a exponentially long proofs.

It suffices to design such a PCP system for the $\mathbf{NP}$-complete problem

$$\mathsf{QUAD} = \{\text{satisfiable systems of quadratic equations over } \mathbb{Z}_2\}.$$

Example 4. An instance of QUAD looks like the following:

$$Q = \begin{cases} x_1x_2 + x_3x_4 & = 1 \\ x_2x_3 & = 0 \\ x_1x_2 + x_2x_4 & = 0. \end{cases}$$

This system is satisfiable, say, with satisfying assignment $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$.

To see this is $\mathbf{NP}$-complete, we can reduce from $\mathsf{CKT}-\mathsf{SAT}$ as follows: Label each gate with a distinct variable, and enforce consistency with the gates feeding into it using a quadratic equation. For instance, if a gate requires $z = x \vee y$, add the constraint $(1-x)(1-y)+z = 1$.

2.1 PCP for QUAD

Random subset sum principle: If $u \neq v \in \mathbb{Z}_2^n$, then

$$\Pr_{x \sim \mathbb{Z}_2^n}[\langle u, x\rangle = \langle v, x\rangle] = \frac{1}{2}.$$

That is, if u and v are distinct bit vectors, then the probability that the XOR of the same random subset of bits from u and from v agree is $1/2$.

Example 5. Let $u = 1011$, $v = 1001$. Then $\langle u, x\rangle = \langle v, x\rangle$ if and only if $x_3 = 0$, which happens with probability $1/2$.

We'll use this principle in a few places in our PCP construction, but the main use is as follows: If u fails to solve a system Q, then it fails to solve a random linear combination of the constraints with probability $1/2$.

The honestly generated proof π for an instance Q of QUAD will consist of the values of all 2^{n^2} quadratic functions of a satisfying assignment to Q. That is,

$$\begin{aligned} \pi &= \left(\sum_{i,j} A_{ij}u_iu_j\right)_{A \in \mathbb{Z}_2^{n\times n}} \\ &= \left(\langle A, uu^\top\rangle\right)_{A \in \mathbb{Z}_2^{n\times n}} \\ &=: \mathrm{WH}(uu^\top) \end{aligned}$$

which is called the "Walsh-Hadamard" encoding of the $n \times n$ matrix $uu^\top$.

Here's the gameplan for probabilistically verifying an alleged proof π^*:

1. Linearity Test: Check (in a manner we'll describe later) that π^* is "close" to $\pi := \mathrm{WH}(vv^\top)$ for some $v \in \mathbb{Z}_2^n$.

2. Random Subset Sum: Take a random linear combination of the constraints in Q to obtain a single quadratic equation $Ax = b$.

3. Local Decoding: Compute $\pi(A)$ using a constant number of probes to π^*, and check that it equals b.

2.2 Linearity Testing

Definition 6. For a vector $v \in \mathbb{Z}_2^m$, the Walsh-Hadamard encoding $\mathrm{WH}(v)$ is the truth table of the linear function $f : \mathbb{Z}_2^m \to \mathbb{Z}_2$ defined by $f(x) = \langle x, v \rangle$. (This is just a 2^m-bit vector.)

In the linearity testing problem, we are given query access to a function $\hat{f} : \mathbb{Z}_2^m \to \mathbb{Z}_2$, and our goal is to test whether it is "close" to some linear function $f(x) = \langle x, v \rangle$.

Note that a function $\hat{f}$ is linear if and only if $\hat{f}(x+y) = \hat{f}(x) + \hat{f}(y)$ for all $x, y \in \mathbb{Z}_2^m$. The BLR (Blum-Luby-Rubinfeld) Test checks the global property of linearity of a function $\hat{f}$ by just checking whether this identity holds for a random pair x, y:

BLR Test: Pick random $x, y \leftarrow \mathbb{Z}_2^m$ and check that $\hat{f}(x+y) = \hat{f}(x) + \hat{f}(y)$.

This test has the following guarantees:

Definition 7. Two functions $f, g : \mathbb{Z}_2^m \to \mathbb{Z}_2$ are δ-close if

$$\Pr_{x \in \mathbb{Z}_2^n}[f(x) = g(x)] \geq 1 - \delta,$$

Completeness: If $\hat{f}$ is linear, then $\Pr[\hat{f}(x+y) = \hat{f}(x) + \hat{f}(y)] = 1$.

Soundness: If $\hat{f}$ is not δ-close to any linear function, then

$$\Pr[\hat{f}(x+y) = \hat{f}(x) + \hat{f}(y)] \leq 1 - \Omega(\delta)$$

Note that this test requires evaluating $\hat{f}$ at only 3 random locations. Moreover, soundness can be amplified through repetition, at the expense of increasing the number of queries to $\hat{f}$.

2.3 Local Decoding

Suppose we know that $\hat{f}$ is δ-close to some linear function f. (Note that if $\delta < 1/4$, then this function f is unique.)

Claim 8. *There is an (efficient) algorithm that computes $f(x)$ (with high probability) using $O(1)$ probes to $\hat{f}$.*

Decoding Procedure: Pick a random $y \leftarrow \mathbb{Z}_2^m$ and compute $\hat{f}(x+y) - \hat{f}(x)$.

Analysis: Observe that for every x, the point $x + y$ (for uniformly random y) is itself uniformly random. Therefore, by a union bound,

$$\begin{aligned}\Pr[\hat{f}(x+y) - \hat{f}(y) \neq f(x)] &\leq \Pr[\hat{f}(x+y) \neq f(x+y)] + \Pr[\hat{f}(y) \neq f(y)] \\ &\leq 2\delta.\end{aligned}$$

2.4 Fixing a Lie

The linearity test we described is able to determine (whp) whether π^* is close to $\mathrm{WH}(M)$ for some $M \in \mathbb{Z}_2^{n\times n}$. But how do we ensure that this M takes the form $uu^\top$ for some $u \in \mathbb{Z}_2^n$?

Solution: We'll enable the verifier to check this by also including an encoding g of u itself as part of the proof.

Test: Given f and g (alleged to encode uu^T and u, respectively), pick $r, s \leftarrow \mathbb{Z}_2^n$ uniformly and test if $f(rs^T) = g(r)g(s)$.

Completeness: If $f = \mathrm{WH}(uu^\top)$ and $g = \mathrm{WH}(u)$, then

$$\begin{aligned} f(rs^\top) &= \sum_{i,j}(rs^\top)_{ij}u_iu_j \\ &= \sum_{i,j}(r_iu_i)(s_ju_j) \\ &= \langle r, u\rangle \cdot \langle s, u\rangle \\ &= g(r)g(s). \end{aligned}$$

Soundness: If $f = \mathrm{WH}(M)$ and $g = \mathrm{WH}(u)$ for some $M \neq uu^\top$, then

$$\begin{aligned} \Pr[f(rs^\top) = g(r)g(s)] &= \Pr[\langle M, rs^\top\rangle = \langle u, r\rangle \cdot \langle u, s\rangle] \\ &= \Pr[\langle s, Mr\rangle = \langle s, uu^\top r\rangle] \\ &= \frac{1}{2} + \frac{1}{2}\Pr[Mr \neq uu^\top r] \\ &= \frac{3}{4}. \end{aligned}$$