

CS 535: Complexity Theory, Fall 2026

Homework 5

Due: 11:59PM, Friday, October 9, 2026.

Reminder. Homework must be typeset with $\text{\LaTeX}$ preferred. Problems 1, 2, 3 (all parts), and 4 are required. Problem 5 is optional.

Problem 1 (*Required* Mostly Existential QBF). Let us say a fully quantified Boolean formula in prenex normal form

$$\Psi = Q_1x_1Q_2x_2\ldots Q_nx_n\varphi(x_1,\ldots,x_n)$$

is “mostly existential” if at most $\log n$ of the quantifiers Q_i are “ $\forall$ ” and the rest are “ $\exists$ ”. (Here, each variable x_i ranges over a single bit.) Show that the language

$$T = \{\Psi \mid \Psi \text{ is a true mostly existential QBF}\}$$

is NP-complete. (As opposed to PSPACE-complete like the ordinary TQBF problem.)

Problem 2 (*Required* Logspace Computability). Let $f : \{0,1\}^* \rightarrow \{0,1\}^*$. Prove that the following two statements are equivalent:

1. The function f is logspace computable in the sense that we defined in class. That is, there is a TM with a read-only input tape and write-only/write-once output tape that computes f using only $O(\log n)$ cells of its (non-input, non-output) work tapes.
2. The function f is implicitly logspace computable in the sense of Arora-Barak. That is, f is polynomially-bounded (i.e., there exists a polynomial p such that $|f(x)| \leq p(|x|)$ for every x) and the languages $\{\langle x, i \rangle \mid i \leq |f(x)|\}$ and $\{\langle x, i \rangle \mid (f(x))_i = 1\}$ are both in L.

Problem 3 (*Required* Time vs. Space). As usual, you can quote anything stated in the main body of Arora-Barak or during the lectures to solve these problems. (Which may give some of them very short solutions.)

- (a) Show that $\text{NL} \subseteq \text{P}$.
- (b) Define $\text{PolyL} = \cup_{c=1}^{\infty} \text{SPACE}(\log^c n)$. Show that $\text{NL} \subseteq \text{PolyL}$.
- (c) Prove that there are no PolyL-complete problems (with respect to logspace reductions). Hint: Assume that such a complete problem were to exist. Show that it would contradict the space hierarchy theorem.
- (d) Define the class SC (“Steve’s Class,” in honor of Stephen Cook) to consist of all languages A for which there exists a deterministic TM M deciding A that runs in time $\text{poly}(n)$ and uses space $\text{poly} \log(n)$. It is an open question whether $\text{NL} \subseteq \text{SC}$. Explain why parts (a) and (b) together do not resolve this open question.

Problem 4 (*Required* NL-Completeness). Define the language DLEN to consist of all tuples $\langle G, s, t, d \rangle$ where G is a directed graph, and the distance from s to t in G is exactly d . (That is, there exists a path from s to t of length d **and there is no shorter path**.) Prove that DLEN is NL-complete.

Hint: To reduce from PATH, make many copies of the original graph vertex set indexed by “time.”

Problem 5 (*Optional* Implication SAT). An “implication CNF” is a CNF where every clause has at most one positive variable. For example, $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1} \vee \overline{x_2})$ is an implication CNF, but $(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1} \vee \overline{x_2})$ is not. The name comes from the (useful) fact that the logical expression $x_1 \vee \overline{x_2} \vee \cdots \vee \overline{x_k}$ is equivalent to $(x_2 \wedge \cdots \wedge x_k) \rightarrow x_1$.

- (a) Define the language $\text{IMPSAT} = \{\varphi \mid \varphi \text{ is a satisfiable implication CNF}\}$. Prove that IMPSAT is P-complete under logspace reductions.
- (b) What goes wrong in your proof if you use it to try to show that IMPSAT is NP-complete?

Hint: You can make the same kinds of simplifying assumptions that we made in our proof sketch of the Cook-Levin Theorem from class—single-tape TM, binary tape alphabet, input padded with trailing zeroes. Focus more on the main ideas than on the details.