

CS 535: Complexity Theory, Fall 2026

Homework 6

Due: 11:59PM, Friday, October 16, 2026.

Reminder. Homework must be typeset with $\text{\LaTeX}$ preferred. Problems 2 (all parts), 3a, and 4a are required. Problem 1 (less challenging but perhaps tedious), Problem 3b (challenging), and Problem 4b (very challenging) are optional.

Problem 1 (*Optional* Alternation Warmup).

- (a) Let $k \in \mathbb{N}$. Prove that a language $L \in \Sigma_2\text{TIME}(n^k)$ if and only if there exists a constant c and a (deterministic) TM $M(x, u, v)$ running in $O(|x|^k)$ steps such that

$$x \in L \iff \exists u \in \{0, 1\}^{c|x|^k} \forall v \in \{0, 1\}^{c|x|^k} M(x, u, v) = 1.$$

- (b) Prove that $\Sigma_2^p = \cup_{k=1}^{\infty} \Sigma_2\text{TIME}(n^k)$.

Problem 2 (*Required* Polynomial Hierarchy).

- (a) Define the language

$$\exists\text{USAT} = \{\varphi \text{ a Boolean formula} \mid \exists x \in \{0, 1\}^n \exists! y \in \{0, 1\}^m \varphi(x_1, \dots, x_n, y_1, \dots, y_m)\}.$$

Here, the notation “ $\exists!$ ” means “there exists exactly one” (satisfying y). For example, $\varphi(x, y_1, y_2) = (x \wedge y_1 \wedge \overline{y_2}) \vee (\overline{x} \wedge y_1) \in \exists\text{USAT}$ because setting $x = 1$ makes $y_1 = 1$ and $y_2 = 0$ the unique satisfying assignment to the formula. Show that $\exists\text{USAT}$ is Σ_2^p -complete.

- (b) Suppose that one day, science shows that $\Sigma_6^p \subseteq \Pi_4^p$. Show that the polynomial hierarchy collapses, to the lowest level that you can.

- (c) Show that if the polynomial hierarchy does not collapse, then $\mathbf{P} \subsetneq \mathbf{PH} \subsetneq \mathbf{PSPACE}$.

Hint: If $\mathbf{PH}$ does not collapse, what can you say about $\mathbf{PSPACE}$ -complete languages?

Problem 3 (Alternation Relations).

- (a) (*Required*) Let $s(n) \geq n$ be space constructible. Prove that $\mathbf{NSPACE}(s(n)) \subseteq \mathbf{ATIME}((s(n))^2)$.

Hint: Recall the proof of Savitch’s Theorem.

- (b) (*Optional*) Let $\mathbf{AL} = \mathbf{ASPACE}(\log n)$ be the class of languages decidable in logarithmic space by an alternating TM. Prove that $\mathbf{P} = \mathbf{AL}$.

Problem 4 (Boolean and Polynomial Hierarchies).

- (a) Define the Boolean hierarchy inductively as follows:

$$\begin{aligned}\mathbf{BH}_1 &= \mathbf{NP} \\ \mathbf{BH}_{2i} &= \{A \cap B \mid A \in \mathbf{BH}_{2i-1}, B \in \mathbf{coNP}\} \\ \mathbf{BH}_{2i+1} &= \{A \cup B \mid A \in \mathbf{BH}_{2i}, B \in \mathbf{NP}\} \\ \mathbf{BH} &= \bigcup_{i=1}^{\infty} \mathbf{BH}_i.\end{aligned}$$

That is, the Boolean hierarchy consists of languages that can be written as a finite combination of unions, intersections, and complements of languages in $\mathbf{NP}$. Show that $\mathbf{BH} \subseteq \mathbf{P}^{\mathbf{NP}} \subseteq \Sigma_2^p \cap \Pi_2^p$.

Hint: $\Sigma_2^p = \mathbf{NP}^{\mathbf{NP}}$.

- (b) (*Mega-optional*) Show that if the Boolean hierarchy collapses, then the polynomial hierarchy collapses.