CS 599B: Math for TCS, Spring 2022

Exercise Set 1

Due: 10:00PM, Monday January 31, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Inner Product)**.** Define the "inner product mod 2 function" $\mathrm{IP}_n : \{-1,1\}^{2n} \to \{-1,1\}$ by $\mathrm{IP}_n(x_1,\dots,x_n,y_1,\dots,y_n) = \prod_{i=1}^n (x_i \wedge y_i)$.

(a) Give the Fourier expansion of IP_n.

(b) Compute the influences ($\mathbf{Inf}_i$) and total influence ($\mathbf{I}$) of IP_n.

(c) Compute the noise stability of IP_n.

Problem 2 (Odd and Even)**.** Let $f : \{-1,1\}^n \to \mathbb{R}$.

(a) Define $f^{\mathrm{odd}}(x) = (f(x) - f(-x))/2$. What are the Fourier coefficients of f^{odd} in terms of the Fourier coefficients of f?

(b) Define $f^{\mathrm{even}}(x) = (f(x) + f(-x))/2$. What are the Fourier coefficients of f^{even} in terms of the Fourier coefficients of f?

(c) Why do you think these functions are called the "odd" and "even" parts of f, respectively?

Problem 3 (Approximation by Low Degree)**.** The *degree* of a function $g : \{-1,1\}^n \to \mathbb{R}$, denoted $\deg(g)$, is the polynomial degree of its Fourier representation, i.e., $\max\{|S| \mid \hat{f}(S) \neq 0\}$. For instance, the degree of $g(x) = x_1x_2x_3 - x_3x_4$ is 3.

Suppose $\mathbf{W}^{>k}[f] := \sum_{|S|>k} \hat{f}(S)^2 \leq \varepsilon$. Show that f is "close" to a low degree function in the sense that there exists a function g with $\deg(g) \leq k$ such that $\mathbb{E}[(f(x) - g(x))^2] \leq \varepsilon$.

Problem 4 (Large coefficients)**.**

(a) Recall that $\mathrm{dist}(f,g) = \Pr_{x\sim\{-1,1\}^n}[f(x) \neq g(x)]$. Verify that $\mathrm{dist}(f,g) = (1 - \langle f,g\rangle)/2$.

(b) Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function. Prove that f can have at most one Fourier coefficient such that $|\hat{f}(S)| > 1/2$.

(c) Is the same statement true for every $f : \{-1,1\}^n \to \mathbb{R}$ for which $\mathbb{E}[f(x)^2] = 1$? Prove it or give a counterexample.