CS 599B: Math for TCS, Spring 2022

Exercise Set 10

Due: 10:00PM, Monday April 25, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (More on expander codes)**. Special note:** You may optionally submit carefully written solutions to this problem to contribute toward your homework grade.

(a) Let C be a $[D_R, k, d]_2$ code. Let $G = ([n], [m], E)$ be a bipartite (α, β)-vertex expander (using the same notation from class) that is D_L-left regular and D_R-right regular. Show that

$$T(G, C) := \{c \in \mathbb{F}_2^n : c|_{N(j)} \in C \quad \forall j \in [m]\}$$

is a linear code. What bounds can you prove on the block length, dimension, and distance? (Under appropriate conditions on α and β.) Here, the notation $c|_{N(j)}$ means the restriction of c to the D_R indices in L that are neighbors of j. Note that in the special case where C is the $[D_R, D_R - 1, 2]_2$ parity check code, this is just the expander code we studied in class.

(b) Let $G = ([n], [m], E)$ be a D-left regular bipartite (α, β)-vertex expander for some $\beta > 19/20$. Show that for some sufficiently large constant C, the following parallel iterative decoder corrects for $\alpha n/4$ errors:

In each of $C \log n$ rounds: do the following in parallel for every bit i of the candidate codeword $\hat{c}$: If $\hat{c}_i$ violates at least $2D/3$ of the constraints it is involved in, flip its value.

Problem 2 (Randomized rounding)**.** In class, we saw an example of how the solution to a linear program exactly solves a combinatorial optimization problem with integrality constraints (bipartite matching). Sometimes we don't get so lucky, but are nevertheless able to use other tricks to convert linear programming solutions to approximately optimal integral ones.

(a) An instance of the MAX-3SAT problem is a collection of clauses $\{\psi_1, \ldots, \psi_m\}$ where each ψ_j is a disjunction of at most 3 literals from $x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}$. The goal is to find an assignment x that (approximately) maximizes the number of satisfied clauses. For each clause ψ_j, let P_j be the set of indices of variables that appear positively in ψ_j, and let N_j be the set of indices of variables that appear negatively. The following is an

LP relaxation for this problem:

$$\begin{aligned}
\max \sum_{j=1}^{m} z_j & & \\
\text{s.t.} \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j & & \forall j = 1, \dots, m \\
0 \leq y_i \leq 1 & & \forall i = 1, \dots, n \\
0 \leq z_j \leq 1 & & \forall j = 1, \dots, m.
\end{aligned}$$

Let (y^*, z^*) be a maximizer for this relaxed LP. Consider the following *randomized rounding* procedure where we independently set each variable x_i to 1 with probability y_i^*. Show that if OPT is the maximum number of satisfiable clauses, then the expected number of satisfied clauses under this randomized rounding strategy is at least $(1 - 1/e)$OPT.

Hint: Show that if clause j contains ℓ_j literals, then the probability that ψ_j is satisfied is at least $1 - (1 - z_j^*/\ell_j)^{\ell_j} \geq (1 - 1/e)z_j^*$.

(b) Now consider the smarter rounding procedure where we independently set each x_i to 1 with probability $y_i^*/2 + 1/4$ and to 0 with the remaining probability. Show that the expected number of satisfied clauses is at least $(3/4)\cdot$OPT.