CS 599B: Math for TCS, Spring 2022

Exercise Set 3

Due: 10:00PM, Monday Feburary 14, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Influence and Noise Sensitivity of PTFs)**.** In this exercise (Exercise 5.45 in O'Donnell), you will show that $\mathbf{I}[f] \leq 2n^{1-2^{-k}}$ whenever $f = \operatorname{sgn}(p)$, where $p : \{-1,1\}^n \to \mathbb{R}$ is a degree-k polynomial that is never zero. The proof is by induction on k. The base case, $k = 0$, is immediate.

(a) Show that for every $i \in [n]$,

$$D_i f = \operatorname{sgn}(D_i p)$$

whenever $(D_i f)(x) \neq 0$.

(b) Use part (a) and then the decomposition $f = x_i D_i f + E_i f$ to show that for every $i \in [n]$,

$$\mathbf{Inf}_i[f] = \mathbb{E}[D_i f(x) \cdot \operatorname{sgn}(D_i p(x)] = \mathbb{E}[f(x) x_i \operatorname{sgn}(D_i p(x))].$$

Hint: Note that $D_i p$ and $E_i p$ do not depend on variable x_i.

(c) Use part (b) to show that

$$\mathbf{I}[f] \leq \mathbb{E}\left[\left|\sum_{i=1}^{n} x_i \operatorname{sgn}(D_i p(x))\right|\right].$$

(d) Apply Cauchy-Schwarz to obtain

$$\mathbf{I}[f] \leq \sqrt{n + \sum_{i \neq j} \mathbb{E}[x_i x_j \operatorname{sgn}(D_i p(x)) \operatorname{sgn}(D_j p(x))]}.$$

(e) Show that for every $i \neq j$, the term

$$\mathbb{E}[x_i x_j \operatorname{sgn}(D_i p(x)) \operatorname{sgn}(D_j p(x))] = \mathbb{E}[D_j \operatorname{sgn}(D_i p(x)) D_i \operatorname{sgn}(D_j p(x))].$$

Hint: This is a special case of the more general statement, which may be easier to reason about, that $\mathbb{E}[x_i x_j f(x) g(x)] = \mathbb{E}[D_j f(x) \cdot D_i g(x)]$ whenever f does not depend on x_i and g does not depend on x_j.

(f) Apply the AGM inequality ($ab \le (a^2 + b^2)/2$) to conclude

$$\mathbf{I}[f] \le \sqrt{n + \sum_{i=1}^{n} \mathbf{I}[\mathrm{sgn}(D_i p)]}.$$

(g) Complete the induction to show that $\mathbf{I}[f] \le 2n^{1-2^{-k}}$.

(h) Use Theorem 5.35 in O'Donnell to conclude that for every $\delta \in (0, 1/2]$, every degree-k PTF has noise sensitivity $\mathbf{NS}_\delta[f] \le O(\delta^{2^{-k}})$.

Problem 2 (Tightness of Bonami)**.**

(a) Show that for $k = 1$, the constant in Bonami's Lemma can be improved to 3. That is, if f is a degree-1 polynomial, then $\mathbb{E}[f(x)^4] \le 3\mathbb{E}[f(x)^2]^2$.

(b) It turns out the constant 9 cannot be improved in general. Consider the degree-k elementary symmetric polynomial

$$f(x) = \sum_{|S|=k} \chi_S(x).$$

First, show that $\mathbb{E}[f(x)^2] = \binom{n}{k}$.

(c) Prove the best lower bound you can on $\mathbb{E}[f(x)^4]$ and compare it to $\binom{n}{k}^2$. For $k \le n/2$ a positive multiple of 3, a lower bound that's true (and which gives a near optimal ratio of $\Theta(9^k/k^2)$) is $\binom{n}{k/3,k/3,k/3,k/3,k/3,k/3,n-2k}$ which is the multinomial coefficient that counts the number of ways to choose six disjoint subsets of $[n]$ each of size $k/3$. But instead of proving that this works when you already know the answer, I think it's more illuminating to explore the problem itself and see what you come up with.