CS 599B: Math for TCS, Spring 2022

Exercise Set 4

Due: 10:00PM, Monday Feburary 21, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Chernoff Bound with Bounded Independence)**.** Let k be an even natural number, and let $X_1, \ldots, X_n \in [-1, 1]$ be k-wise independent random variables, each with mean zero. Let $\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$. Show that

$$\Pr[|\overline{X}| \geq t] \leq \frac{(k/2)^k}{n^k t^k} \cdot \binom{n}{k/2} \leq \left(\frac{k^2}{4nt^2}\right)^{k/2}.$$

Hint: Compute the kth moment, and observe that every nonzero term in its expansion has to involve at least two factors of each X_i it depends on.

Problem 2 (General Constructions of k-Wise Independence)**.**

a) Let $\mathbb{F}$ be a finite field and let $Y \in \mathbb{F}^{n \times k}$ be any matrix such that every subset of k of the rows of Y are linearly independent. Let $\vec{r} = Y\vec{a} \in \mathbb{F}^n$ where $\vec{a} \leftarrow \mathbb{F}^k$ is uniformly random. Show that the random variables $(r_1, \ldots, r_n)$ are k-wise independent.

b) What is the matrix Y corresponding to the construction of k-wise independent random variables we saw in class? If you want, you can try to show that every square submatrix of this matrix has full rank.

Problem 3 (Ramsey Graphs in Quasi-Polynomial Time)**.**

a) An undirected graph on n vertices is k-Ramsey if it does *not* contain a clique or independent set of size k. Show that, for every n, there exists a k-Ramsey graph for $k = O(\log n)$. Hint: Use the probabilistic method. Consider a random graph where every edge is included independently with probability $1/2$. Show that with positive probability, such a random graph is $O(\log n)$-Ramsey.

b) Derandomize your construction using $\binom{k}{2}$-wise independence. Use this to construct a deterministic algorithm that, given an integer n, constructs an $O(\log n)$-Ramsey graph in time $n^{O(\log^2 n)}$.