CS 599B: Math for TCS, Spring 2022

Exercise Set 6

Due: 10:00PM, Monday March 28, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Cycle)**.** Let C_n be the undirected cycle on n vertices: $V = [n]$ and $E = \{(i, i+1) \mid 1 \le i \le n-1\} \cup \{(n, 1)\}$.

(a) Show that the conductance of C_n is $\phi(C_n) = \Theta(1/n)$.

(b) Let ν_2 be the second eigenvalue of the normalized Laplacian $N = \frac{1}{2}L$. Show that $\nu_2 = O(1/n^2)$. Hint: Apply the variational characterization of ν_2 to an appropriate choice of a test function $f \perp \mathbf{1}$.

Thus, this is an example for which the right-hand side (the "hard direction") of Cheeger's inequality is tight.

(c) Optional problems: Show that the bound in part (b) is tight: $\nu_2 = \Theta(1/n^2)$. Show that $\nu_k = \Theta(k/n^2)$ for every $k \ge 2$.

Problem 2 (Spectrum of the Normalized Laplacian)**.** Let N be the normalized Laplacian of an undirected graph G. Show that the largest eigenvalue satisfies $\nu_n \le 2$. Hint: Show that for every $f \ne 0$, we have $\langle f, Lf \rangle \le \langle f, Df \rangle$.

Optional: When is the upper bound of 2 attained?