CS 599B: Math for TCS, Spring 2022

Exercise Set 7

Due: 10:00PM, Monday April 4, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Adding self-loops)**.** Let G be a d-regular graph on n vertices. Consider the $2d$-regular graph $\widetilde{G}$ obtained by adding d self-loops to every vertex of G.

(a) What are the eigenvalues of the normalized Laplacian $N_{\widetilde{G}}$ in terms of the eigenvalues of N_G?

(b) Show that if G is an $(n/2, \varepsilon)$-edge expander, then $\widetilde{G}$ is a $(\varepsilon^2/4)$-spectral expander.

Problem 2 (Spectral approximation)**.** Show that if G is a γ-spectral expander on n vertices, then $\|N_G - N_K\| \leq 1 - \gamma$, where K is the complete graph with self-loops on n vertices.

Hint: If $f \perp \mathbf{1}$, what can you say about $N_K f$?

Problem 3 (ℓ_2 convergence of an expander walk)**.** Let W be the walk matrix of a d-regular γ-spectral expander (V, E). Let p_0 be an arbitrary starting distribution on V and let π be the uniform distribution. Note that π is the stationary distribution of W. Consider the distribution p_t of the t'th step of a random walk starting from p_0. Show that

$$\|p_t - \pi\|_2^2 = \|W^t(p_0 - \pi)\|_2^2 \leq (1 - \gamma)^{2t}\|p_0 - \pi\|_2^2.$$