Instructions. You’re encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don’t worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Adding self-loops). Let $G$ be a $d$-regular graph on $n$ vertices. Consider the $2d$-regular graph $\tilde{G}$ obtained by adding $d$ self-loops to every vertex of $G$.

(a) What are the eigenvalues of the normalized Laplacian $N_{\tilde{G}}$ in terms of the eigenvalues of $N_G$?

(b) Show that if $G$ is an $(n/2, \varepsilon)$-edge expander, then $\tilde{G}$ is a $(\varepsilon^2/4)$-spectral expander.

Problem 2 (Spectral approximation). Show that if $G$ is a $\gamma$-spectral expander on $n$ vertices, then $\|N_G - N_K\| \leq 1 - \gamma$, where $K$ is the complete graph with self-loops on $n$ vertices.

Hint: If $f \perp \mathbf{1}$, what can you say about $N_K f$?

Problem 3 ($\ell_2$ convergence of an expander walk). Let $W$ be the walk matrix of a $d$-regular $\gamma$-spectral expander $(V, E)$. Let $p_0$ be an arbitrary starting distribution on $V$ and let $\pi$ be the uniform distribution. Note that $\pi$ is the stationary distribution of $W$. Consider the distribution $p_t$ of the $t$'th step of a random walk starting from $p_0$. Show that

$$\|p_t - \pi\|_2^2 = \|W^t(p_0 - \pi)\|_2^2 \leq (1 - \gamma)^{2t}\|p_0 - \pi\|_2^2.$$