CS 599B: Math for TCS, Spring 2022

Exercise Set 8

Due: 10:00PM, Monday April 11, 2022 on Gradescope.

Instructions. You're encouraged to discuss these problems with other students as you solve them, but the written solutions you hand in must be in your own words. Don't worry about polishing the presentation of your solutions; these are primarily intended to keep you thinking about and engaged with the material.

Problem 1 (Loewner Order)**.** Recall that for symmetric PSD matrices A and B, we define $A \preceq B$ if $B - A$ is PSD.

(a) Show that if $A \preceq B$, then $CAC^T \preceq CBC^T$ for any matrix C.

(b) Let G and H be graphs such that $w_e^G \leq w_e^H$ for every $e \in V \times V$. (That is, G is a subgraph of H with all smaller edge weights.) Show that $L_G \preceq L_H$.

(c) Suppose A and B are simultaneously diagonalizable: there exists an orthonormal basis $\psi_1, \dots, \psi_n$ of eigenvectors for both A and B. Show that $A \preceq B$ if and only if $\lambda_i(A) \leq \lambda_i(B)$ for every $i = 1, \dots, k$, where λ_i is the eigenvalue corresponding to ψ_i. Does such a characterization hold without the assumption of a simultaneous eigenbasis?

Problem 2 (Rayleigh's Monotonicity Principle)**.** Let G be a weighted graph with weighted Laplacian L determined by $\langle f, Lf \rangle = \sum_{(a,b)\in E} w_{a,b}(f(a) - f(b))^2$.

(a) Show that for every pair of vertices a, b, we have

$$\frac{1}{R_{\text{eff}}(a,b)} = \min_{f : f(a)=1, f(b)=0} \langle f, Lf \rangle.$$

Hint: Let $v = L^+(\mathbf{1}_a - \mathbf{1}_b)$. Use calculus to show that $f = \frac{v - v(b)\mathbf{1}}{R_{\text{eff}}(a,b)}$ is the unique minimizer of the convex function on the RHS.

(b) Use part (a) to show that if G and H are graphs such that $w_e^G \leq w_e^H$ for every $e \in V \times V$, then $R_{\text{eff}}^G(e) \geq R_{\text{eff}}^H(e)$ for every $e \in V \times V$. This is Rayleigh's monotonicity principle: If you add resistors to a network, then the effective resistances can't go down.