CS 599B: Math for TCS, Spring 2022

Homework 3

Due: 10:00PM, Friday April 29, 2022 on Gradescope.

Instructions. Solutions must be typeset, with LaTeX strongly preferred. You are encouraged to collaborate on the homework problems with each other in small groups (2-4 people). Collaboration may include brainstorming or exploring possible solutions together on a whiteboard, but should not include one person telling the others how to solve a problem. You must also write up the solutions independently (in your own words) and acknowledge your collaborators at the beginning of the first page.

You may freely use without proof any results proved in class, in Mark's lecture notes posted on the class webpage, or in the main body of the texts assigned as reading. Note that this excludes results that appear in the texts as problems and exercises. You may, of course, use such results but you have to prove them first.

Problem 1 (Diameter vs. second eigenvalue)**.**

a) Let G be any connected, undirected, unweighted graph with diameter r. Show that for every $f \perp \mathbf{1}$,

$$\min_{f \neq 0, f \perp \mathbf{1}} \frac{\langle f, Lf \rangle}{\langle f, f \rangle} \geq \Omega(1/rn)$$

and conclude that $\lambda_2(G) \geq \Omega(1/rn)$.

b) Let G be a γ-spectral expander for $\gamma \geq 1/2$. Show that G has diameter at most $O(\log n)$.

Problem 2 (Tightness of convergence rate)**.** Let G be a d-regular undirected graph with lazy walk matrix $\widetilde{W}$ having second eigenvalue $\omega_2 \geq 1/2$. In class, we showed that lazy walks on such graphs mix quickly. In this problem, you will show that the mixing time upper bound we proved is tight up to logarithmic factors.

Show that for $\varepsilon > 0$, there exists a vertex s such that if p_t is the length-t lazy random walk starting at vertex s, then

$$TV(p_t, \pi) \geq \varepsilon$$

unless $t \geq \Omega(\log(1/\varepsilon)/(1-\omega_2))$.

Problem 3 (Random walks and effective resistances)**.** Let G be a connected, undirected, unweighted graph. Think of G as a resistor network where every edge present in the graph has resistance 1. Let $m = |E|$ be the number of edges in G. For vertices a and b, define the *hitting time* $H_{a,b}$ from a to b to be the expected number of steps a random walk starting at a needs to take before reaching b. Note that $H_{b,b} = 0$.

(a) Show that for every $a \neq b$, we have $H_{a,b} = \sum_{u \sim a} \frac{1}{\deg(a)}(1 + H_{u,b})$.

(b) Consider an external current defined by $i_{\text{ext}}(u) = \deg(u)$ for every $u \in V \setminus \{b\}$ and $i_{\text{ext}}(b) = \deg(b) - 2m$. You can think of this as injecting $\deg(u)$ units of current into every vertex and having all of the current flowing out of vertex b. Let $v_b(a)$ denote the voltage induced at a when i_{ext} is applied, defining $v_b(b) = 0$. Show that for every $a \neq b$, we have $v_b(a) = \sum_{u \sim a} \frac{1}{\deg(a)}(1 + v_b(u))$.

(c) Show that for every a, b, we have $H_{a,b} = v_b(a)$. Hint: Argue that the linear system satisfied by both functions has a unique solution.

(d) Show that for every a, b, we have

$$H_{a,b} + H_{b,a} = 2mR_{\text{eff}}(a, b).$$