CAS CS 599 B: Mathematical Methods for TCS

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Lecture Notes 13:

Intro to spectral graph theory, Laplacian and its eigenvalues

Reading.

- Trevisan Chapter 3
- Spielman Chapters 2 and 3

Spectral graph theory relates the combinatorial properties of graphs (e.g., connectivity, colorability, mixing of random walks) to the linear algebraic properties of associated matrices. It's a central tool in both algorithm design and complexity, with applications to:

- Markov chains. In a (large) graph where random walks mix quickly, one can sample efficiently sample from the stationary distribution by simulating a random walk.
- Expanders, and thereby to derandomization.
- Spectral sparsification for the design of fast graph algorithms.
- Solving systems of linear equations.
- Finding electrical flows.

1 Combinatorial Laplacian

Let $G = (V, E)$ be an undirected graph with $n = |V|$. Let $f : V \to \mathbb{R}$ be a function on the vertices of G. An important quantity in spectral graph theory is the "Laplacian quadratic form":

$$Q[f] := \sum_{(u,v)\in E} (f(u) - f(v))^2.$$

How should we think about this quantity? Suppose f is the 0-1 indicator for a subset S of vertices. Recalling our notation $\partial S = \{(u, v) \in E \mid u \in s, v \notin S\}$, we have that $Q[\mathbf{1}_S] = |\partial S|$.

The name for this quantity comes from the fact that it's the quadratic form associated to the "Laplacian" linear operator. This is the operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Lf(u) = \deg(u)f(u) - \sum_{v\sim u} f(v).$$

Claim 1.

$$Q[f] = \langle f, Lf\rangle = \sum_{u\in V} f(u) \cdot Lf(u).$$

Proof. We can rewrite the LHS as

$$\sum_{(u,v)\in E} f(u)^2 - 2f(u)f(v) + f(v)^2 = \sum_{u\in V} f(u)^2 \deg(u) - \sum_{u\in V}\sum_{v\sim u} f(u)f(v),$$

which is equal to the RHS. $\square$

The Laplacian operator has a nice matrix representation. Let D be the *degree* operator $Df(u) := \deg(u)f(u)$, which is specified by the matrix

$$D = \begin{pmatrix} \deg(u_1) & 0 & \dots & 0 \\ 0 & \deg(u_2) & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \deg(u_n) \end{pmatrix}.$$

Let A be the *adjacency matrix* of the graph G, specified by $A[u,v] = 1$ if $(u,v) \in E$ and $A[u,v] = 0$ otherwise. Then $L = D - A$.

2 Eigenvalues of the Laplacian

Spectral graph theory is all about understanding how the algebraic properties of matrices like L and A tell us combinatorial information about G. So let's start investigating some properties of the eigenvalues of L.

Definition 2. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator (matrix). A function (vector) $f : V \to \mathbb{R}$ is an eigenvector of M with eigenvalue λ if $Mf = \lambda f$.

Fact 3. *The constant function* $\mathbf{1}$ *is an eigenvector of* L *with eigenvalue* 0.

Fact 4. *The eigenvalues of* L *are all nonnegative, i.e.,* L *is positive semidefinite.*

Proof. The lazy way to see this is to note that the Laplacian quadratic form $Q[f]$ is always nonnegative, so L is PSD. For a slightly more hands-on explanation, suppose $Lf = \lambda f$. Then by Claim 1, $Q[f] = \sum_{u\in V} \lambda f(u)^2 \geq 0$ which implies $\lambda \geq 0$. $\square$

Theorem 5. *Sort the eigenvalues of a Laplacian* L *in nondecreasing order:* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. *Then* $\lambda_2 = 0$ *iff* G *is disconnected.*

More generally, $\lambda_k = 0$ iff G has at least k connected components.

Proof. First suppose G is disconnected. Let S and T be two disjoint connected components. Then $L\mathbf{1}_S = 0$ and $L\mathbf{1}_T = 0$, so L as two linearly independent eigenvectors of eigenvalue 0. Hence $\lambda_2 = 0$.

For the other direction, we prove the contrapositive. Suppose G is connected and f is an eigenvector of L with eigenvalue 0. We claim that f must be a scalar multiple of $\mathbf{1}$. To see this, observe that

$$0 = Q[f] = \sum_{(u,v)\in E} (f(u) - f(v))^2.$$

This implies that $f(u) = f(v)$ for all neighboring u, v. Since G is connected, we must have $f(u) = f(v)$ for all vertices u, v, which means f is in the subspaced spanned by $\mathbf{1}$. $\square$

Example 6. Let K_n be the complete graph with self-loops. It has adjacency matrix

$$A = J_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Its Laplacian is

$$L = nI_n - J_n = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & & & \\ -1 & -1 & \dots & n-1 \end{pmatrix}.$$

As with any Laplacian matrix, this has $\lambda_1 = 0$. We claim that $\lambda_2 = \cdots = \lambda_n = n$. This will follow from the fact that for any function f that is orthogonal to $\mathbf{1}$, we have $Lf = nf$. To see this, we compute for any $u \in V$:

$$Lf(u) = nf(u) - \sum_{(u,v)\in E} f(v) = nf(u) - \sum_{v \in V} f(v) = nf(u),$$

where the last equality holds because $f \perp \mathbf{1}$ means $\sum_{v\in V} f(v) = 0$.

3 Rayleigh Quotient

The smallest "interesting" eigenvalue of a Laplacian is the second one. Theorem 5 tells us that it characterizes when G is connected. It turns out that when $\lambda_2 > 0$, its magnitude quantifies how connected G is.

Here's some intuition. Suppose again that $f = \mathbf{1}_S$ is the 0-1 indicator for a set S of vertices, and suppose G is d-regular. That is, every vertex in G has exactly d neighbors. (This assumption can be lifted using a more complicated definition of L.) Recall our interpretation that $Q[f] = |\partial S|$ measures the boundary size of S. We can interpret the ratio

$$\frac{|\partial S|}{|S|} = \Pr_{(u,v)\in E}[v \notin S | u \in S].$$

Now consider that

$$|S| = \sum_{u \in V} f(u)^2.$$

So the ratio

$$\frac{|\partial S|}{|S|} = \frac{\langle f, Lf \rangle}{\langle f, f \rangle}$$

tells us something about how likely a random point in S is to have a neighbor outside of S.

Definition 7. Given an operator M and a function f, define the Rayleigh quotient

$$R_M[f] := \frac{\langle f, Mf \rangle}{\langle f, f \rangle}.$$

To complement the combinatorial interpretation above, we describe a close relationship between the Rayleigh quotient and the eigenvalues of M. First observe that if f is an eigenvalue of M with eigenvalue λ, then $R_M[f] = \lambda$. This suggests a way to compute the eigenvalues of M by peeling them off one-by-one in the following sense.

Theorem 8. *If M is a symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding eigenvectors $f_1, \ldots, f_n$. Then*

$$\lambda_1 = \min_{f \neq 0} R_M[f],$$

$$\lambda_2 = \min_{f \neq 0, f \perp f_1} R_M[f], \ldots$$

$$\lambda_k = \min_{f \neq 0, f \in S_{k-1}^{\perp}} R_M[f]$$

where $S_{k-1} = \mathrm{span}\{f_1, \ldots, f_{k-1}\}$

Proof sketch. By the spectral theorem for real symmetric matrices, we know that we can take the eigenvectors $f_1, \ldots, f_n$ to be orthogonal. Thus, the condition $f \in S_{k-1}^{\perp}$ implies $f \perp f_i$ for all $i = 1, \ldots, k-1$. So for $f \in S_{k-1}^{\perp}$, we have

$$\langle \sum_{i=k}^{n} c_i f_i, M \sum_{i=k}^{n} c_i f_i \rangle = \langle \sum_{i=k}^{n} c_i f_i, \sum_{i=k}^{n} \lambda_i c_i f_i \rangle = \sum_{i=k}^{n} \lambda_i c_i^2 \geq \lambda_k \sum_{i=k}^{n} c_i^2.$$

The denominator of $R_M[f]$, on the other hand, is $\sum_{i=k}^{n} c_i^2$. So the Rayleigh quotient itself is always at least λ_k. Meanwhile, plugging in $f = f_k$ itself yields $R_M[f_k] = \lambda_k$. $\square$