CAS CS 599 B: Mathematical Methods for TCS

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Lecture Notes 17:

More on Expanders, Resistor Networks

Reading.

- Vadhan Section 4.3, Spielman 11.7-11.9, 12.1-12.4

Recall the definition of a spectral expander:

Definition 1 (Spectral Expansion)**.** A graph G is a γ-spectral expander if $\nu_2 \geq \gamma$ and $(2 - \nu_n) \geq \gamma$, where $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \leq 2$ are the eigenvalues of the normalized Laplacian $N = \frac{1}{d}L = I - \frac{1}{d}A$.

1 Extractors from Expanders

Random walks on expanders also provide a useful method for constructing an extractor. Intuitively, if one starts a random walk from a vertex chosen from a weak random source, and then takes a short random walk, then one ends up at a nearly uniform vertex. We will now make this precise. Recall:

Definition 2. A (k, ε)-seeded extractor is a function $\mathrm{Ext} : \{0,1\}^n \times \{0,1\}^s \to \{0,1\}^m$ such that $TV(\mathrm{Ext}(X, \mathcal{U}_s), \mathcal{U}_m) \leq \varepsilon$ for every distribution X over $\{-1,1\}^n$ with min-entropy $\geq k$.

Suppose we have a strongly explicit family of d-regular γ-spectral expanders, with $\gamma \geq 1/2$. Consider the following algorithm $\mathrm{Ext} : \{0,1\}^n \times \{0,1\}^{t \log d} \to \{0,1\}^n$. On input $x \in \{0,1\}^n$ and $y \in \{0,1\}^{t \log d}$, interpret y as a sequence $y_1, \dots, y_t \in [d]$ of directions to take. Starting at x in a graph G on vertex set $\{0,1\}^n$, follow these directions and output the destination vertex.

Theorem 3. *For $t = (n-k)/2 + \log(1/\varepsilon)$, the algorithm* Ext *described above is a (k, ε)-seeded extractor.*

Proof. Let X be a k-source, let p_0 be the start distribution of the random walk, and let p_t be the distribution of the destination vertex. Letting π denote the uniform distribution on V, we have

$$\begin{aligned}
\|p_t - \pi\|_2^2 &= \|W^t(p_0 - \pi)\|_2^2 \\
&\leq (1-\gamma)^{2t} \|p_0 - \pi\|_2^2 \\
&\leq 2^{-2t} \cdot (C(p_0) - 2^{-n}) \\
&\leq 2^{-2t} \cdot (2^{-k} - 2^{-n}) \\
&\leq 2^{-2t-k}.
\end{aligned}$$

Here, we recall that the collision probability $C(p)$ is defined by $\sum_v p(v)^2$.

The first inequality is a consequence of spectral expansion that you proved as an exercise. The second uses the fact that $\|p - \pi\|^2 = C(p) - 2^{-n}$, which we proved as part of the proof of Lemma 9 in Lecture 12. The third uses the fact that $C(p) \leq 2^{-k}$ for any distribution p with min entropy at least k.

Now using these facts in reverse, and setting $t = (n-k)/2 + \log(1/\varepsilon)$ we get

$$C(p_t) = \|p_t - \pi\|_2^2 + 2^{-n} \le \frac{1+\varepsilon^2}{2^n}$$

Using Lemma 9 from Lecture 12, which says that $TV(p_t, \pi) \le 2^{n/2-1}\sqrt{C(p_t) - 2^{-n}}$, we get that $TV(p_t, \pi) \le \varepsilon$. $\square$

2 Limits of Spectral Expansion

How good can the spectral expansion of a d-regular graph be? The following gives a lower bound:

Proposition 4. *For every d-regular graph of diameter at least 4, we have $\nu_2 \le 1 - 1/\sqrt{d}$.*

Proof. Choose two vertices u and v whose neighborhoods have no edges between them, i.e., there are no edges between $N(u)$ and $N(v)$ where $N(u) = \{s \mid s \sim u\}$. Define the test function

$$g(x) = \begin{cases} 1 & \text{if } x = u \\ 1/\sqrt{d} & \text{if } x \in N(u) \\ -1 & \text{if } x = v \\ -1/\sqrt{d} & \text{if } x \in N(v) \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the Rayleigh quotient of L at f. We have

$$\begin{aligned} Q[g] &= 2d(1 - 1/\sqrt{d})^2 + 2d(d-1)(1/\sqrt{d})^2 \\ &= 2(2d - 2\sqrt{d}). \end{aligned}$$

Meanwhile,

$$\langle g, g \rangle = 2 + 2d \cdot (1/\sqrt{d})^2 = 4.$$

Hence we have

$$\nu_2 = \min_{f \ne 0, f \perp \mathbf{1}} \frac{\langle f, Nf \rangle}{\langle f, f \rangle} \le \frac{Q[g]}{d\langle g, g \rangle} = 1 - \frac{1}{\sqrt{d}}.$$

$\square$

Some additional work yields the Alon-Boppana bound:

Theorem 5. *For every d and every infinite family $\{G_n\}$ of d-regular graphs,*

$$\nu_2 \le 1 - \frac{2\sqrt{d-1}}{d} + \varepsilon(n)$$

where $\varepsilon(n) \to 0$ as $n \to \infty$.

3 Constructions of Expanders

We'd like to construct infinite sequences of d-regular expander graphs which, for algorithmic applications, should be strongly explicit. Unfortunately we won't be able to do justice to any of the constructions, but here is a rundown of the known techniques for constructing expanders and pointers you can look into.

Probabilistic method. Theorem 4.4 in Vadhan illustrates how to show that random graphs are good combinatorial expanders. A random d-regular graph matches the Alon-Boppana bound with high probability. This was conjectured by Noga Alon around 1986, and proved by Friedman in 2004: `https://arxiv.org/abs/cs/0405020`.

Algebraic constructions. In 1973, Margulis gave the first explicit construction of (a continuous) analog of expanders; the discrete version was analyzed by Gabber and Gallai in 1981. Later, Lubotzky, Phillips, and Sarnak and, independently, Margulis, gave explicit constructions of *Ramanujan graphs* when $d = p+1$ for a prime p that is congruent to 1 modulo 4. Ramanujan graphs are graphs which meet the Alon-Boppana bound *without the $\varepsilon(n)$ term.*

Combinatorial constructions. Reingold, Vadhan, and Wigderson introduced a natural framework for constructing expanders by starting with a small expander on a constant number of vertices, and repeatedly applying graph operations to obtain an expander on a larger number of vertices. Let an (n, d, γ)-graph have n vertices, degree d, and spectral expansion γ. Consider interleaving the following operations:

Squaring An an edge between u and v in G' iff they are connected by a path of length 2 in G. Then G' is an $(n, d^2, 2\gamma - \gamma^2)$-graph.

Tensoring On vertex set $V \times V$, connect (u_1, u_2) to (v_1, v_2) iff $(u_1, v_1) \in E$ and $(u_2, v_2) \in E$. Then G' is an (n^2, d^2, γ)-graph.

Zig-Zag This is a more complex operation that takes an (n, d, γ)-graph G and a small fixed (n', d', γ')-graph H and creates a $(nn', (d')^2, \gamma(\gamma')^2)$-graph G'.

Note that each operation improves one parameter while making the other two worse. You can see the details of these operations and how to piece them together in Chapter 4.3 of Vadhan. For a simpler construction of constant-degree expander that illustrates a lot of the same ideas, see Chapter 30 of Spielman.

4 Resistor Networks

Let G be an undirected *weighted* graph. Think of G as a resistor network, where each edge e is a resistor with resistance r_e. Given a *voltage* function $v : V \to \mathbb{R}$ on the vertices, Ohm's Law ($V = IR$) implies that the current from a to b is

$$i(a, b) = \frac{v(a) - v(b)}{r_{(a,b)}} = w_{(a,b)} \cdot (v(a) - v(b)).$$

Note that even though the graph is undirected, this is a directed quantity: $i(a, b) = -i(b, a)$. The net current from a to the rest of the network is

$$i_{ext}(a) = \sum_{b \in V} w_{(a,b)} \cdot (v(a) - v(b)) = (Lv)(a)$$

where L is the usual Laplacian matrix, but taking into account edge weights.

The principle of conservation of flow implies the net current flowing in an out of a vertex must be 0. So if $i_{ext}(a) \neq 0$, that means some external current is being applied at a. One can think of the vertices for which $i_{ext}(a) \neq 0$ as "boundary vertices" through which current is entering or exiting the network, and the other vertices as "internal vertices."

Let's think about what it means for $i_{ext} \equiv 0$. This corresponds to a voltage function for which $Lv = 0$, which for a connected graph G, occurs if and only if v is a constant function.

Now let's consider a more general version of this problem. Given a vector of external currents i_{ext}, what are the induced voltages on the vertices in the graph? Algebraically, this is a solution v to the linear system

$$Lv = i_{ext}.$$

This system has problems owing to the fact that L is singular. Nevertheless, we have:

1. $Lv = i_{ext}$ has a solution iff $i_{ext} \perp \mathbf{1}$, i.e., the net current flowing into the network is zero.
2. Solutions to the system are unique up to constants. That is, v is a solution if and only if $v + c\mathbf{1}$ is a solution for every $c \in \mathbb{R}$.

A canonical solution to this system is given by the Moore-Penrose pseudo-inverse of L. The general definition is kind of complicated, but for symmetric matrices, it's characterized as follows:

Definition 6. The pseudo-inverse of a symmetric matrix L, written L^+, is the matrix such that

1. $\mathrm{Im}(L^+) = \mathrm{Im}(L)$, and
2. $LL^+ = \Pi$ where Π is the symmetric matrix that projects onto $\mathrm{Im}(L)$.

As usual, it's easiest to think about L^+ in terms of the eigendecomposition of L. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ be the eigenvalues of L with corresponding orthonormal eigenvectors $\psi_1 = \mathbf{1}, \psi_2, \ldots, \psi_n$. Then we can write

$$L = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^T = \sum_{i=2}^{n} \lambda_i \psi_i \psi_i^T.$$

The pseudoinverse is given by

$$L^+ = \sum_{i=2}^{n} \frac{1}{\lambda_i} \psi_i \psi_i^T.$$

So given a linear system $Lv = i_{ext}$ where $i_{ext} \perp \mathbf{1}$, we can take the canonical solution $v = L^+ i_{ext}$. This is the "balanced" solution, i.e., the one such that $v \perp \mathbf{1}$.

5 Effective Resistance

Fix two vertices a, b in a resistor network. We can think of the network as a one large resistor connecting a and b. What is the resistance of this equivalent resistor?

Recalling Ohm's Law, $i(a,b) = (v(a) - v(b))/r_{(a,b)}$, we can define the effective resistance $R_{\text{eff}}(a,b) = v(a) - v(b)$ where v is the vector of induced potentials when one unit of current enters the network at a and exits at b. That is, v is the solution to

$$Lv = \mathbf{1}_a - \mathbf{1}_b,$$

i.e., $v = L^+(\mathbf{1}_a - \mathbf{1}_b)$ and we are interested in $v(a) - v(b)$. Thus,

$$R_{\text{eff}}(a,b) = \langle \mathbf{1}_a - \mathbf{1}_b, L^+(\mathbf{1}_a - \mathbf{1}_b) \rangle = \|L^{+/2}\mathbf{1}_a - L^{+/2}\mathbf{1}_b\|^2$$

where $L^{+/2}$ is the matrix square root of positive semidefinite L^+, explicitly given by

$$L^{+/2} = \sum_{i=2}^{n} \frac{1}{\sqrt{\lambda_i}} \psi_i \psi_i^T$$

using the decomposition above.