

An Isomorphism Extension Theorem For Landau-Ginzburg B-Models

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Abstract

Landau-Ginzburg mirror symmetry studies isomorphisms between A- and B-models, which are graded Frobenius algebras that are constructed using a weighted homogeneous polynomial W and a related symmetry group G . Given two polynomials W_1, W_2 with the same weights and same group G , the corresponding A-models built with (W_1, G) and (W_2, G) are isomorphic. Though the same result cannot hold in full generality for B-models, which correspond to orbifolded Milnor rings, we provide a partial analogue. In particular, we exhibit conditions where isomorphisms between unorbifolded B-models (or Milnor rings) can extend to isomorphisms between their corresponding orbifolded B-models (or orbifolded Milnor rings).

1 Introduction

Landau-Ginzburg mirror symmetry studies two different physical theories, known as Landau-Ginzburg A- and B-models, which are graded Frobenius algebras that are built using a nondegenerate weighted homogeneous polynomial W and a related group of symmetries G of W . The A-model theories (denoted by \mathcal{A}) have been constructed (Fan et al., 2013), and are a special case of what is known as *FJRW theory*. The B-model theories (denoted by \mathcal{B}) have also been constructed (Intriligator and Vafa, 1990; Kaufmann, 2002, 2003, 2006; Krawitz, 2010), and correspond to an *orbifolded Milnor ring*. In many cases, these theories extend to whole families of Frobenius algebras, called *Frobenius manifolds*.

For a large class of polynomials, Berglund and Hübsch (1993), Berglund and Henningson (1995), and Krawitz (2010) described the construction of a dual (or transpose) polynomial W^T and a dual group G^T . The Landau-Ginzburg mirror symmetry conjecture states that the A-model of a pair (W, G) should be isomorphic to the B-model of the dual pair (W^T, G^T) , and is denoted as $\mathcal{A}[W, G] \cong \mathcal{B}[W^T, G^T]$. This conjecture has been proven in many cases (Francis et al., 2012; Krawitz, 2010), although the proof of

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the full conjecture remains open. To better understand mirror symmetry, it has been fruitful to focus on studying isomorphisms between Landau-Ginzburg models of the same type: either from A to A, or from B to B.

The Landau-Ginzburg A-model is *deformation invariant* (see Tay, 2013). Given two polynomials with the same weights, and an admissible symmetry group that fixes both polynomials, there exists a continuous path to deform one polynomial to the next. All the corresponding A-models along such a path are isomorphic as graded Frobenius algebras—this result is sometimes called the *Group-Weights Theorem*. The same result does not hold for B-models (see Example 2.25).

The unorbifolded Landau-Ginzburg B-model, which is built using the trivial group G , corresponds to the Milnor ring (or local algebra) of a polynomial W and is often denoted as \mathcal{Q}_W . The original construction of the vector space structure of the orbifolded Milnor ring, or orbifolded B-model, was given by Intriligator and Vafa (1990). The product structure remained undefined for many years. Recently, Krawitz (2010) followed ideas presented by Kaufmann (2002, 2003, 2006) to write down a multiplication for the orbifolded Milnor ring.

Classical singularity theory has widely studied Milnor rings of polynomials and their related isomorphisms. In this paper, we look at providing a partial Group-Weights result for orbifolded Milnor rings. That is, we look at extending known isomorphisms between Milnor rings to isomorphisms between orbifolded Milnor rings that have the product structure defined by Krawitz (2010). We will often refer to these orbifolded Milnor rings as Landau-Ginzburg B-models.

We approach this problem by focusing on special choices of polynomials and groups. Let W be an *admissible* polynomials (see Definition 2.3), and let G be a subgroup of the *maximal symmetry group* of W (see Definition 2.7). Building on ideas presented in Francis et al. (2012), we arrive at the following conditions for a polynomial/group pair.

Definition 1.1. A pair (W, G) is *well behaved* if $W = \sum W_i$, where each W_i is an admissible polynomial in distinct variables, and $G = \bigoplus G_i$, where each $g \in G_i$ either fixes all or none of the variables of W_i for each i .

As we will note later, a large class of polynomial/group pairs that satisfy this condition include the two-variable admissible polynomials together with any of their symmetry groups. However, some polynomials in three or more variables (such as *chain* polynomials) may have choices of symmetry groups that do not form well-behaved pairs.

We note that Definition 1.1 is similar to Property (*) of Francis et al. (2012) (see also Definition 3.1), which gives conditions that guarantee a mirror symmetry isomorphism between an A-model with (W, G) and its dual B-model with (W^T, G^T) . We also require that the particular isomorphism between Milnor rings be *equivariant*. That is, when applying a nontrivial group of symmetries, the isomorphism respects the group action on the Milnor ring’s vector space basis. The following theorem is the main result of the

paper.

Theorem 3.6. *Let (W, G) and (V, G) be well behaved. If $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ is an equivariant isomorphism of graded Frobenius algebras, then ϕ extends to an isomorphism $\psi : \mathcal{B}[W, G] \rightarrow \mathcal{B}[V, G]$.*

It turns out that when Milnor rings are isomorphic, the corresponding polynomials will have the same weights (up to ordering of variables, see Theorem 2.22). Therefore Theorem 3.6 is a partial Group-Weights theorem for orbifolded Milnor rings. In Section 4 we give examples of cases where this theorem applies.

2 Preliminaries

Here we will introduce some of the concepts needed to understand the theory of this paper.

2.1 Admissible Polynomials

Definition 2.1. For a polynomial $W \in \mathbb{C}[x_1, \dots, x_n]$, we say that W is *nondegenerate* if it has an isolated critical point at the origin.

Definition 2.2. Let $W \in \mathbb{C}[x_1, \dots, x_n]$. We say that W is *quasihomogeneous* if there exist positive rational numbers q_1, \dots, q_n such that for any $c \in \mathbb{C}$, $W(c^{q_1}x_1, \dots, c^{q_n}x_n) = cW(x_1, \dots, x_n)$.

We often refer to the q_i as the *quasihomogeneous weights* of a polynomial W , or just simply the *weights* of W , and we write the weights in vector form $J = (q_1, \dots, q_n)$.

Definition 2.3. $W \in \mathbb{C}[x_1, \dots, x_n]$ is *admissible* if W is nondegenerate and quasihomogeneous with unique weights, having no monomials of the form $x_i x_j$ for $i \neq j$, $i, j \in \{1, \dots, n\}$.

The condition that W have no cross-term monomials is necessary for constructing the A-model (see Fan et al., 2013). Because the construction of $\mathcal{A}[W, G]$ requires an admissible polynomial, we will only be concerned with admissible polynomials in this paper. In order for a polynomial to be admissible, it needs to have at least as many monomials as variables. Otherwise its quasihomogeneous weights cannot be uniquely determined. We now state the main subdivision of the admissible polynomials.

Definition 2.4. Let W be an admissible polynomial. We say that W is *invertible* if it has the same number of monomials as variables. If W has more monomials than variables, then it is *noninvertible*.

Admissible polynomials with the same number of variables as monomials are called invertible, since their associated exponent matrices are square and invertible.

Definition 2.5. Let $W \in \mathbb{C}[x_1, \dots, x_n]$. If we write W as a sum of monomials $W = \sum_{i=1}^m c_i \prod_{j=1}^n x_j^{a_{ij}}$, then the associated *exponent matrix* is defined to be $A = (a_{ij})$.

We further observe that if W is invertible, we can rescale variables to force each coefficient c_i to equal one—which we will do in this paper. The invertible polynomials can also be decomposed into sums of three types of polynomials, called the *atomic types*.

Theorem 2.6 (Theorem 1 of Kreuzer and Skarke (1992)). *Any invertible polynomial is the decoupled sum of polynomials in one of three atomic types:*

$$\begin{aligned} \text{Fermat type: } & W = x^a, \\ \text{Loop type: } & W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1, \\ \text{Chain type: } & W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}. \end{aligned}$$

We also assume that the $a_i \geq 2$ to avoid terms of the form $x_i x_j$ for $i \neq j$.

2.2 Symmetry Groups

Definition 2.7. Let W be an admissible polynomial. We define the *maximal diagonal symmetry group* of W to be $G_W^{\max} = \{(\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^\times)^n \mid W(\zeta_1 x_1, \dots, \zeta_n x_n) = W(x_1, \dots, x_n)\}$.

The proofs of Lemma 2.1.8 in Fan et al. (2013) and Lemma 1 in Artebani et al. (2014) show that G_W^{\max} is finite and that each coordinate of every group element is a root of unity. The group operation in G_W^{\max} is coordinate-wise multiplication. But since additive notation is often more convenient, we use the map $(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n}) \mapsto (\theta_1, \dots, \theta_n) \pmod{\mathbb{Z}}$ taking G_W^{\max} to $(\mathbb{Q}/\mathbb{Z})^n$. Hence we will often write $G_W^{\max} = \{g \in (\mathbb{Q}/\mathbb{Z})^n \mid Ag \in \mathbb{Z}^m\}$, where A is the $m \times n$ exponent matrix of W . In this notation we have the following

Definition 2.8. The group G_W^{\max} is a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$ with respect to coordinate-wise addition. For $g \in G_W^{\max}$, we can write g uniquely as (g_1, \dots, g_n) , where each g_i is a rational number in the interval $[0, 1)$. The g_i are called the *phases* of g .

That being said, as a matter of convenience we will often use equivalent representatives of the g_i that lie outside the interval $[0, 1)$ to write down group elements.

2.3 Graded Frobenius Algebras

Landau-Ginzburg A- and B-models are algebraic objects that are endowed with many levels of structure. In this paper, we will chiefly be concerned with their structure up to the level of graded Frobenius algebras. We will only develop the theory needed for this paper. We refer the interested reader to Fan et al. (2013) for more details on the construction of the A-model. Francis et al. (2012), Krawitz (2010), and Tay (2013) also contain more information on constructing A- and B-models, and related isomorphisms.

Definition 2.9. A *graded Frobenius algebra* is a graded algebra A with a *pairing* $\langle \cdot, \cdot \rangle : A \times A \rightarrow \mathbb{C}$ that is

- Symmetric: $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in A$,
- Bilinear: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in A$ and $\alpha, \beta \in \mathbb{C}$,
- Nondegenerate: for every $x \in A$ there exists $y \in A$ such that $\langle x, y \rangle \neq 0$.

The pairing further satisfies the *Frobenius property*, meaning that $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$ for all $x, y, z \in A$ with $x \neq 0$.

2.4 Unorbifolded B-Models

Definition 2.10. For any polynomial W , the algebra $\mathcal{Q}_W = \mathbb{C}[x_1, \dots, x_n] / (\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n})$ is called the *Milnor ring* (or *local algebra*) of W .

We note that \mathcal{Q}_W has a vector space structure with a basis consisting of monomials.

Theorem 2.11 (Theorem 2.6 of Tay (2013)). *If W is admissible, then \mathcal{Q}_W is finite dimensional.*

We will further think of the Milnor ring as a graded vector space over \mathbb{C} , by defining the degree of a monomial in \mathcal{Q}_W to be $\deg(x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}) = 2 \sum_i^n a_i q_i$, where the q_i are the quasihomogeneous weights of W . We have the following well-known results about the vector space structure of the Milnor ring (see Section 2.1 of Krawitz (2010)). First, $\dim(\mathcal{Q}_W) = \prod_{i=1}^n \left(\frac{1}{q_i} - 1\right)$. Second, the highest degree of its graded pieces is $2 \sum_{i=1}^n (1 - 2q_i)$. The number $\sum_{i=1}^n (1 - 2q_i)$ is called the *central charge*, and is denoted by \hat{c} .

To make \mathcal{Q}_W into a graded Frobenius algebra, we need to define a pairing function.

Definition 2.12. Let W be an admissible polynomial W , and let $m, n \in \mathcal{Q}_W$. We define the *pairing* $\langle m, n \rangle$ to be the complex number that satisfies

$$mn = \frac{\langle m, n \rangle}{\mu} \text{Hess}(W) + \text{terms of degree less than } \deg(\text{Hess}(W)),$$

where μ is the dimension of \mathcal{Q}_W as a vector space, and $\text{Hess}(W)$ is the *Hessian* of W which is the determinant of the matrix of second partial derivatives of W .

As noted by Krawitz (2010), we can represent $\text{Hess}(W)$ as a monomial in the Milnor ring. Further, the elements of highest degree in the Milnor ring form a one-dimensional subspace that is spanned by $\text{Hess}(W)$.

The Milnor ring, together with the grading of the monomial basis and this pairing function, forms a graded Frobenius algebra (for details, reference Arnold et al.).

Definition 2.13. We define the *unorbifolded B-model* $\mathcal{B}[W, \{\mathbf{0}\}]$ by $\mathcal{B}[W, \{\mathbf{0}\}] =: \mathcal{Q}_W$.

2.5 Orbifolded B-Models

Constructing orbifolded B-models for a general group G has historically been a hard problem. Kaufmann did a lot of work in this area (Kaufmann, 2002, 2003, 2006), but in this paper we focus on the most important case for Landau-Ginzburg mirror symmetry: the diagonal subgroup of $\mathrm{SL}(n, \mathbb{C})$. We follow the construction of Krawitz (2010), built on the work of Kaufmann.

Definition 2.14. Let $W \in \mathbb{C}[x_1, \dots, x_n]$ be admissible. Let $g \in G_W^{\max}$, writing g in multiplicative coordinates. The *fixed locus* of the group element g is the subspace of \mathbb{C}^n that is fixed by g . Using coordinate-wise multiplication, we write $\mathrm{fix}(g) = \{x \in \mathbb{C}^n \mid g(x) = x\}$.

The notation $W|_{\mathrm{fix}(g)}$ denotes the restriction of the polynomial W to the domain $\mathrm{fix}(g)$. We now define the group action of G on the Milnor ring.

Definition 2.15. Let W be an admissible polynomial, and let $g \in G_W^{\max}$. In multiplicative coordinates, write $g = \mathrm{diag}[\zeta_1, \dots, \zeta_n]$. For a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ we define the map $g^* : \mathcal{Q}_W \rightarrow \mathcal{Q}_W$ by $g^*(m) := \det(g)m \circ g = \det(g)(\zeta_1 x_1)^{a_1} \cdots (\zeta_n x_n)^{a_n}$, and extend linearly. This is the *group action* on the elements of \mathcal{Q}_W , sometimes denoted as $g \cdot m$.

Definition 2.16. Let W be an admissible polynomial, and let $G \leq G_W^{\max}$. The *G -invariant subspace* of \mathcal{Q}_W is defined to be $\mathcal{Q}_W^G = \{m \in \mathcal{Q}_W \mid g^*(m) = m \text{ for each } g \in G\}$.

To construct an orbifolded B-model, we restrict G to be a subgroup of $G_W^{\max} \cap \mathrm{SL}(n, \mathbb{C})$.

Definition 2.17. Let W be an admissible polynomial, and $G \leq G_W^{\max} \cap \mathrm{SL}(n, \mathbb{C})$ where n is the number of variables of W . We define the underlying vector space of $\mathcal{B}[W, G]$ to be $\bigoplus_{g \in G} \left(\mathcal{Q}_W|_{\mathrm{fix}(g)} \right)^G$, where $(\cdot)^G$ denotes all the G -invariants. This is called the B-model *state space*.

The condition that $G \leq G_W^{\max} \cap \mathrm{SL}(n, \mathbb{C})$ is required to construct the orbifolded B-model. We will often denote the group $G_W^{\max} \cap \mathrm{SL}(n, \mathbb{C})$ as $\mathrm{SL}(W)$.

Note that if we let $G = \{\mathbf{0}\}$, then the formula yields the Milnor ring of W , as expected. We also note that the vector space basis of $\mathcal{B}[W, G]$ is made up of monomials from the basis of $\mathcal{Q}_W|_{\mathrm{fix}(g)}$ for each $g \in G$. We denote these basis elements $[m; g]$, where g is a group element and m is a monomial in $\left(\mathcal{Q}_W|_{\mathrm{fix}(g)} \right)^G$.

To make $\mathcal{B}[W, G]$ into a graded Frobenius algebra, we will define the grading, the multiplication and the pairing function. We'll start with the vector space grading.

Definition 2.18. Let W be an admissible polynomial with weights (q_1, \dots, q_n) . For a basis element $[m; (g_1, \dots, g_n)]$ in the vector space basis for $\mathcal{B}[W, G]$, we define its *degree* to be

$$2p + \sum_{g_i \notin \mathbb{Z}} (1 - 2q_i),$$

where p is the weighted degree of m . That is, if $m = x_1^{a_1} \cdots x_n^{a_n}$, then $p = \sum_{i=1}^n a_i q_i$.

The definition of B-model multiplication is due to Krawitz (2010), Kaufmann (2002, 2003, 2006), and Intriligator and Vafa (1990).

Definition 2.19. The product of two elements $[m; g]$ and $[n; h]$ is given by

$$[m; g] \star [n; h] = \begin{cases} [\gamma nm; g+h] & \text{if } \text{fix}(g) \cup \text{fix}(h) \cup \text{fix}(g+h) = \mathbb{C}^n \\ 0 & \text{otherwise} \end{cases}$$

where γ is a monomial defined as

$$\gamma = \frac{\mu_{g \cap h} \text{Hess}(W|_{\text{fix}(g+h)})}{\mu_{g+h} \text{Hess}(W|_{\text{fix}(g) \cap \text{fix}(h)})}.$$

Here $\mu_{g \cap h}$ is the dimension of the Milnor ring of $W|_{\text{fix}(g) \cap \text{fix}(h)}$, and μ_{g+h} is the dimension of the Milnor ring of $W|_{\text{fix}(g+h)}$.

We note that Krawitz proved this multiplication to be associative in the case that W is an invertible polynomial (see Proposition 2.1 of Krawitz (2010)). We believe this to also always be associative when W is noninvertible polynomial, but it has never been proven in general.

Finally, we have the pairing function.

Definition 2.20. Let $[m; g]$ and $[n; h]$ be two basis elements of $\mathcal{B}[W, G]$. If $g = -h$, then $\mathcal{Q}_{W|_{\text{fix}(g)}}$ is canonically isomorphic to $\mathcal{Q}_{W|_{\text{fix}(h)}}$. Therefore, we can define the *pairing* on $\mathcal{B}[W, G]$ as follows:

$$\langle [m; g], [n; h] \rangle = \begin{cases} \langle m, n \rangle_{\mathcal{Q}_{W|_{\text{fix}(g)}}} & \text{if } g = -h, \\ 0 & \text{otherwise.} \end{cases}$$

One can verify that the orbifolded B-model $\mathcal{B}[W, G]$, as it has been defined, is a graded Frobenius algebra (see Section 2.4 of Krawitz, 2010).

2.6 Isomorphisms of Graded Frobenius Algebras

We will focus on studying isomorphisms between Landau-Ginzburg B-models. The following are some common results about isomorphisms between unorbifolded B-models. We will refer back to these later on in the paper. Note that we consider two polynomials to be equivalent if they define the same singularity at the origin. That is, we say that $f \sim g$ if there exists a diffeomorphism $h : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $f = g \circ h$.

Theorem 2.21 (Theorem 2.2.8 of Suggs (2012)). *If W_1 and W_2 are quasihomogeneous functions fixing the origin, then W_1 and W_2 are equivalent if and only if their Milnor rings are isomorphic.*

Theorem 2.22 (Theorem 5.1.1 of Suggs (2012)). *If two nondegenerate quasihomogeneous polynomials are equivalent, they have the same unordered set of weights.*

Theorem 2.23 (Webb's Theorem, Theorem 5.1.3 of Suggs (2012)). *Let W_1 and W_2 be nondegenerate quasihomogeneous polynomials with the same (ordered) weights. If no elements in \mathcal{Q}_{W_1} have weighted degree 1, then W_1 and W_2 are equivalent.*

These are all results about B-model isomorphisms using the trivial group $\{\mathbf{0}\}$. The following result includes orbifolded B-models.

Proposition 2.24 (Proposition 2.3.2 of Francis et al. (2012)). *Suppose W_1 and W_2 are nondegenerate, quasihomogeneous polynomials with no variables in common. If $G_1 \leq SL(W_1)$ and $G_2 \leq SL(W_2)$, then $G_1 \times G_2$ is contained in $SL(W_1 + W_2)$, $G_1 \times G_2$ fixes $W_1 + W_2$, and we have an isomorphism*

$$\mathcal{B}[W_1, G_1] \otimes \mathcal{B}[W_2, G_2] \cong \mathcal{B}[W_1 + W_2, G_1 \times G_2].$$

Note that Theorem 2.23 is a type of Group-Weights result on the B-side. However, Group-Weights does not hold in general for B-models as the next example demonstrates.

Example 2.25 (Example 5.1.4 of Suggs (2012)). Let $W_1 = x^4 + y^4$ and $W_2 = x^3y + xy^3$. Both polynomials have weights $(\frac{1}{4}, \frac{1}{4})$. The set $\{1, y, y^2, x, xy, xy^2, x^2, x^2y, x^2y^2\}$ is a basis for both \mathcal{Q}_{W_1} and \mathcal{Q}_{W_2} . One can verify that any ring homomorphism from \mathcal{Q}_{W_1} to \mathcal{Q}_{W_2} will not be surjective, so we see that $\mathcal{B}[W_1, \{\mathbf{0}\}] \not\cong \mathcal{B}[W_2, \{\mathbf{0}\}]$. But notice that x^2y^2 has weighted degree 1. We see that any choice of basis for \mathcal{Q}_{W_1} or \mathcal{Q}_{W_2} will contain a monomial of weighted degree 1. Therefore this does not contradict Webb's Theorem.

This shows that Group-Weights is not sufficient for B-model isomorphisms. This also shows that deformation invariance does not hold in general on the B-side, since there is no way to deform $x^4 + y^4$ into $x^3y + xy^3$ while maintaining isomorphic Milnor rings.

3 Isomorphism Extension Theorem

Though the Group-Weights theorem does not hold in general for B-models, we still want to find instances where it does. So given equivalent singularities W_1, W_2 with a common group $G \leq SL(n, \mathbb{C})$ that fixes them both, we want to find cases when their corresponding B-models $\mathcal{B}[W_1, G]$ and $\mathcal{B}[W_2, G]$ are also isomorphic. We will need to impose a condition on our polynomials and groups, which condition in part stems from the following definition. Note that the *sector* of an A- or B-model corresponding to a group element g refers to the subset of the vector space basis containing the elements of the form $[m; g]$.

Definition 3.1 (Property (*) of Francis et al. (2012)). Let W be a nondegenerate, invertible polynomial, and let G be an admissible group of symmetries of W . The pair (W, G) has *Property (*)* if

1. W can be decomposed as $W = \sum_{i=1}^M W_i$, where the W_i are themselves invertible polynomials having no variables in common with any other W_j , $j \neq i$.
2. For any element g of G whose associated sector $\mathcal{A}_g \subseteq \mathcal{A}[W, G]$ is nonempty, and for each $i \in \{1, \dots, M\}$ the action of g fixes either all of the variables in W_i or none of them.
3. For any element g' of G^T whose associated sector of $\mathcal{B}_{g'} \subseteq \mathcal{B}[W^T, G^T]$ is nonempty, and for each $i \in \{1, \dots, M\}$ the action of g' fixes either all of the variables in W_i^T or none of them.

Property (*) in Francis et al. (2012) is a generalization of the *well behaved* condition for a polynomial/group pair (W, G) given in Definition 1.1. We note that for the following polynomials, any possible choice of group (that fixes the polynomial and is contained in $\mathrm{SL}(n, \mathbb{C})$) will form a well-behaved pair: fermats, loops in any number of variables, and any admissible polynomial in two variables. We can further admit arbitrary sums of fermat and loop polynomials in distinct variables, together with any of their symmetry groups (see Remark 1.1.1 of Francis et al. (2012)).

Theorem 3.2. *Let W_1 and W_2 be admissible polynomials with $\phi : \mathcal{Q}_{W_1} \rightarrow \mathcal{Q}_{W_2}$ an equivariant isomorphism of graded Frobenius algebras, and let G be a group that preserves both W_1 and W_2 . If (W_1, G) and (W_2, G) are well behaved, then ϕ extends to an isomorphism $\psi : \mathcal{B}[W_1, G] \rightarrow \mathcal{B}[W_2, G]$.*

Consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{B}[W_1, G] & \xrightarrow{\psi} & \mathcal{B}[W_2, G] \\
 \uparrow & & \uparrow \\
 \mathcal{B}[W_1, \{\mathbf{0}\}] & \xrightarrow{\phi} & \mathcal{B}[W_2, \{\mathbf{0}\}]
 \end{array}$$

The bottom horizontal arrow is the isomorphism we are given by hypothesis. The dashed vertical arrows point from each unorbifolded B-model to its corresponding orbifolded B-model. In general, there is no isomorphism going from bottom to top. The top horizontal arrow is the map that is conjectured to exist. In essence, we want to take the map ϕ that we are given, and use it to create an isomorphism of orbifolded B-models.

Proof. By hypothesis, there exists an equivariant isomorphism $\phi : \mathcal{Q}_{W_1} \rightarrow \mathcal{Q}_{W_2}$. Also by hypothesis, we'll assume that ϕ is equivariant with respect to G . Suppose that a monomial basis for \mathcal{Q}_{W_1} is $\mathrm{span}_{\mathbb{C}}\{m_1 = 1, \dots, m_k\}$. We obtain a basis for \mathcal{Q}_{W_2} with $\mathrm{span}_{\mathbb{C}}\{\phi(m_1) = 1, \dots, \phi(m_k)\}$.

Suppose that $(\mathcal{Q}_{W_1})^G = \mathrm{span}_{\mathbb{C}}\{p_1, \dots, p_l\}$, where each $p_i = m_j$ for some j , and $l \leq k$. Since ϕ is equivariant, we have that $g \cdot \phi(p_i) = \phi(g \cdot p_i) = \phi(p_i)$. Therefore $\mathrm{span}_{\mathbb{C}}\{\phi(p_1), \dots, \phi(p_l)\} \subseteq (\mathcal{Q}_{W_2})^G$. But

if we take an m_i not preserved under the action of G , we get $g \cdot \phi(m_i) = \phi(g \cdot m_i) = \phi(cm_i) = c\phi(m_i)$ for some constant $c \neq 1$. Therefore $(\mathcal{Q}_{W_2})^G = \text{span}_{\mathbb{C}}\{\phi(p_1), \dots, \phi(p_l)\}$.

Notice that the same process works even if we first restrict W_1 to a fixed locus of a group element. So for $(\mathcal{Q}_{W_1|_{\text{fix}(g)}})^G$, we can write it as $\text{span}_{\mathbb{C}}\{r_i\}$ where the r_i form a subset of the m_i . We see that $(\mathcal{Q}_{W_2|_{\text{fix}(g)}})^G = \text{span}_{\mathbb{C}}\{\phi(r_i)\}$ as before. This gives us the following: there are (not necessarily distinct) group elements h_1, \dots, h_l such that

$$\begin{aligned}\mathcal{B}[W_1, G] &= \text{span}_{\mathbb{C}}\{[p_1; h_1], \dots, [p_l; h_l]\}, \\ \mathcal{B}[W_2, G] &= \text{span}_{\mathbb{C}}\{[\phi(p_1); h_1], \dots, [\phi(p_l); h_l]\}.\end{aligned}$$

Now define the map $\psi : \mathcal{B}[W_1, G] \rightarrow \mathcal{B}[W_2, G]$ by $\psi([p_i; h_i]) = [\phi(p_i); h_i]$. Notice that ψ is a well-defined bijection that preserves the vector space grading. Also ψ maps the identity $[1; \mathbf{0}]$ to the identity $[1; \mathbf{0}]$.

That ψ preserves the pairing is also easy to show. Let $B_1 = \mathcal{B}[W_1, G]$ and $B_2 = \mathcal{B}[W_2, G]$. Using the properties of pairings, we have for $h_i + h_j = \mathbf{0}$,

$$\langle [p_i; h_i], [p_j; h_j] \rangle_{B_1} = \langle p_i, p_j \rangle_{\mathcal{Q}_{W_1}} = \langle \phi(p_i), \phi(p_j) \rangle_{\mathcal{Q}_{W_2}} = \langle [\phi(p_i); h_i], [\phi(p_j); h_j] \rangle_{B_2}.$$

Since all other pairings are zero, this shows that ψ respects the pairing.

Now for the products. For basis elements α, β of B_1 , we want to show that $\psi(\alpha \star \beta) = \psi(\alpha) \star \psi(\beta)$. We'll consider the case where $\text{fix}(h_i) \cup \text{fix}(h_j) \cup \text{fix}(h_i + h_j) = \mathbb{C}^n$. Otherwise, both products will be zero. First,

$$\psi(\alpha \star \beta) = \psi([p_i; h_i] \star [p_j; h_j]) = \psi([\gamma_1 p_i p_j; h_i + h_j]) = [\phi(\gamma_1 p_i p_j); h_i + h_j] = [\phi(\gamma_1) \phi(p_i p_j); h_i + h_j].$$

The last equality comes from considering γ_1 as a monomial in \mathcal{Q}_{W_1} . Here we have

$$\gamma_1 = \frac{\mu_{h_i \cap h_j} \text{Hess}(W_1|_{\text{fix}(h_i + h_j)})}{\mu_{h_i + h_j} \text{Hess}(W_1|_{\text{fix}(h_i) \cap \text{fix}(h_j)}}.$$

Second, we have

$$\psi(\alpha) \star \psi(\beta) = [\phi(p_i); h_i] \star [\phi(p_j); h_j] = [\gamma_2 \phi(p_i) \phi(p_j); h_i + h_j] = [\gamma_2 \phi(p_i p_j); h_i + h_j].$$

Here we have

$$\gamma_2 = \frac{\mu_{h_i \cap h_j} \text{Hess}(W_2|_{\text{fix}(h_i + h_j)})}{\mu_{h_i + h_j} \text{Hess}(W_2|_{\text{fix}(h_i) \cap \text{fix}(h_j)}}.$$

Previously, we computed bases for the Milnor rings of W_1 and W_2 after restricting to fixed loci and taking G -invariants. Since the dimension remained the same between W_1 and W_2 after these operations, we see that $\mu_{h_i \cap h_j}$ for W_1 equals $\mu_{h_i \cap h_j}$ for W_2 and similarly for $\mu_{h_i + h_j}$. So it just remains to check how ϕ deals with the respective Hessians. That is, we will have $[\phi(\gamma_1)\phi(p_i p_j); h_i + h_j] = [\gamma_2 \phi(p_i p_j); h_i + h_j]$ if we can show $\phi(\gamma_1) = \gamma_2$. We'll consider the behavior of group elements, and break this down into cases.

Case 1: $h_i = h_j = \mathbf{0}$. Notice that W_i restricted to the fixed locus is just W_i again. So the Hessians divide each other, which shows that $\gamma_1 = \gamma_2$. Further, $\mu_{h_i \cap h_j} = \mu_{h_i + h_j}$, which shows that $\gamma_1 = \gamma_2 = 1$. Therefore $\phi(\gamma_1) = \gamma_2$.

Case 2: one of $h_i, h_j = \mathbf{0}$. Without loss of generality, $h_i = \mathbf{0}$. So $\gamma_1 = \frac{\mu_{h_j} \text{Hess}(W_1|_{\text{fix}(h_j)})}{\mu_{h_j} \text{Hess}(W_1|_{\text{fix}(h_j)})} = 1$. Similarly, $\gamma_2 = 1$. Therefore $\phi(\gamma_1) = \gamma_2$.

Case 3: Both h_i, h_j are nonzero. By hypothesis on the behavior of our group elements, we will have the fixed locus of h_i and h_j trivial. But $h_i + h_j$ must be $\mathbf{0}$ in order to get a nonzero product. Therefore $\gamma_1 = \frac{\text{Hess}(W_1)}{\mu}$, $\gamma_2 = \frac{\text{Hess}(W_2)}{\mu}$. We will have $\phi(\gamma_1) = \gamma_2$ if we can show that $\phi(\text{Hess}(W_1)) = \text{Hess}(W_2)$. This is taken care of by Lemma 3.3, which verifies Case 3 and finishes the proof of the theorem. \square

Lemma 3.3. *If $\phi : \mathcal{B}[W_1, \{\mathbf{0}\}] \rightarrow \mathcal{B}[W_2, \{\mathbf{0}\}]$ is an isomorphism of B -models, then $\phi(\text{Hess}(W_1)) = \text{Hess}(W_2)$.*

Proof. Let $B_1 = \mathcal{B}[W_1, \{\mathbf{0}\}]$ and $B_2 = \mathcal{B}[W_2, \{\mathbf{0}\}]$. Suppose m_1, m_2 are monomials in the basis of B_1 such that $m_1 m_2$ spans the sector of highest degree in B_1 . Since ϕ is an isomorphism, we can write $B_2 = \text{span}_{\mathbb{C}}\{\phi(m) \mid m \text{ is a basis element of } B_1\}$. Also, we know that ϕ preserves pairings:

$$\langle m_1, m_2 \rangle_{B_1} = \langle \phi(m_1), \phi(m_2) \rangle_{B_2}.$$

Recall that $m_1 m_2 = \frac{\langle m_1, m_2 \rangle_{B_1}}{\mu} \text{Hess}(W_1)$, where $\mu = \dim(B_1)$. Since $B_1 \cong B_2$, we also have that $\mu = \dim(B_2)$. Now note that $\text{Hess}(W_1) = \frac{\mu(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}}$. Apply ϕ :

$$\phi(\text{Hess}(W_1)) = \phi\left(\frac{\mu(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}}\right) = \frac{\mu \phi(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}} = \frac{\mu \phi(m_1 m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}.$$

On the other hand, we know by the isomorphism that the element $\phi(m_1 m_2) = \phi(m_1)\phi(m_2)$ spans the sector of highest degree in B_2 . We have that $\phi(m_1)\phi(m_2) = \frac{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}{\mu} \text{Hess}(W_2)$. So then

$$\text{Hess}(W_2) = \frac{\mu \phi(m_1)\phi(m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}} = \frac{\mu \phi(m_1 m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}.$$

This shows that $\phi(\text{Hess}(W_1)) = \text{Hess}(W_2)$, as desired. \square

We can now generalize the result to sums of polynomials.

Corollary 3.4. *Let $W = W_1 + W_2$ and $V = V_1 + V_2$ be sums of admissible polynomials in distinct variables where $\phi_i : \mathcal{Q}_{W_i} \rightarrow \mathcal{Q}_{V_i}$ is an equivariant isomorphism of graded Frobenius algebras for each i . If (W_i, G_i) and (V_i, G_i) form well-behaved pairs for each i , then there exists an isomorphism $\psi : \mathcal{B}[W, G] \rightarrow \mathcal{B}[V, G]$ where $G = G_1 \times G_2$.*

Proof. First we'll construct an isomorphism $\phi : \mathcal{B}[W, \{\mathbf{0}\}] \rightarrow \mathcal{B}[V, \{\mathbf{0}\}]$ using the ϕ_i .

Claim: By the tensor product structure (see Proposition 2.24), we know that any monomial m_i in the basis of \mathcal{Q}_W can be written as $\alpha_i \beta_i$ where the α_i is in the basis of \mathcal{Q}_{W_1} and the β_i is in the basis of \mathcal{Q}_{W_2} . We can define ϕ by $\phi : m_i \mapsto \phi_1(\alpha_i) \phi_2(\beta_i)$ and extend linearly.

Proof of Claim: It is easy to verify that ϕ is a bijection, is linear, sends the identity to the identity, and preserves degrees. To show that ϕ respects the pairing, we note that

$$\begin{aligned} \langle \phi(m_i), \phi(m_j) \rangle_{\mathcal{Q}_V} &= \langle \phi_1(\alpha_i) \phi_2(\beta_i), \phi_1(\alpha_j) \phi_2(\beta_j) \rangle_{\mathcal{Q}_V} \\ &= \langle \phi_1(\alpha_i), \phi_1(\alpha_j) \rangle_{\mathcal{Q}_{V_1}} \langle \phi_2(\beta_i), \phi_2(\beta_j) \rangle_{\mathcal{Q}_{V_2}} \\ &= \langle \alpha_i, \alpha_j \rangle_{\mathcal{Q}_{W_1}} \langle \beta_i, \beta_j \rangle_{\mathcal{Q}_{W_2}} \\ &= \langle \alpha_i \beta_i, \alpha_j \beta_j \rangle_{\mathcal{Q}_W} \\ &= \langle m_i, m_j \rangle_{\mathcal{Q}_W}. \end{aligned}$$

For the products, we note that

$$\begin{aligned} \phi(m_i m_j) &= \phi(\alpha_i \beta_i \alpha_j \beta_j) = \phi(\alpha_i \alpha_j \beta_i \beta_j) = \phi_1(\alpha_i \alpha_j) \phi_2(\beta_i \beta_j) = \phi_1(\alpha_i) \phi_1(\alpha_j) \phi_2(\beta_i) \phi_2(\beta_j) \\ &= \phi_1(\alpha_i) \phi_2(\beta_i) \phi_1(\alpha_j) \phi_2(\beta_j) = \phi(\alpha_i \beta_i) \phi(\alpha_j \beta_j) = \phi(m_i) \phi(m_j). \end{aligned}$$

Therefore ϕ really is an isomorphism of graded Frobenius algebras. We further check that ϕ is equivariant: for $g \in G$, we have $g \cdot \phi(m) = g \cdot (\phi_1(\alpha) \phi_2(\beta)) = (g \cdot \phi_1(\alpha))(g \cdot \phi_2(\beta))$, since α and β are in distinct variables, $= \phi_1(g \cdot \alpha) \phi_2(g \cdot \beta)$, since ϕ_1 and ϕ_2 are equivariant, $= \phi(g \cdot m)$.

Now given our map ϕ , we see that W and V are equivalent singularities. Construct map ψ as before, but with using ϕ as the base map. The only thing left to check is that ψ respects products for group elements with nontrivial fixed locus. First note that with the W_i in distinct variables, the block matrix structure of the second partial derivatives of W will give us $\text{Hess}(W) = \text{Hess}(W_1) \text{Hess}(W_2)$. It follows that ϕ sends $\text{Hess}(W_i)$ to $\text{Hess}(V_i)$ by Lemma 3.3 and by construction. Now the group elements g, h have to fix all the variables in either W_1 or W_2 by the hypothesis of the symmetry group structure. In this way any quotient of Hessians will reduce to either $\text{Hess}(W_1)$ or $\text{Hess}(W_2)$. This shows that ψ respects the products, and gives us the desired isomorphism. \square

We now include a brief result on equivariant isomorphisms.

Lemma 3.5. *Suppose (W, G) and (V, G) are well behaved. Then an isomorphism $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ is equivariant if and only if we have equivariant isomorphisms $\phi_i : \mathcal{Q}_{W_i} \rightarrow \mathcal{Q}_{V_i}$ for each i .*

Proof. (\Rightarrow) Suppose that $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ is an equivariant isomorphism of graded Frobenius algebras. We can write $W = W_1 + \cdots + W_n$ and $V = V_1 + \cdots + V_n$ where each W_i is in the same variables as V_i but W_i is in distinct variables from W_j for all $i \neq j$. We can also write $G = G_1 \times \cdots \times G_n$, where G_i preserves either all or none of the variables of W_i, V_i for each i . By Proposition 2.24, we can consider $\mathcal{Q}_W \cong \mathcal{Q}_{W_1} \otimes \cdots \otimes \mathcal{Q}_{W_n}$ and $\mathcal{Q}_V \cong \mathcal{Q}_{V_1} \otimes \cdots \otimes \mathcal{Q}_{V_n}$. From the tensor product structure, we find that there exists a basis of each \mathcal{Q}_{W_i} that is a subset of a basis of \mathcal{Q}_W . By restricting ϕ to the variables of W_i , we obtain an equivariant isomorphism $\phi_i : \mathcal{Q}_{W_i} \rightarrow \mathcal{Q}_{V_i}$ for each i .

(\Leftarrow) Conversely, suppose that we have equivariant isomorphisms $\phi_i : \mathcal{Q}_{W_i} \rightarrow \mathcal{Q}_{V_i}$ for each i . The argument in the proof of Corollary 3.4 shows how to construct an equivariant isomorphism $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ in the case that $n = 2$. Extending by induction gives us the result for all n . \square

We are now ready to obtain the main result of the paper.

Theorem 3.6. *Let (W, G) and (V, G) be well behaved. If $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ is an equivariant isomorphism of graded Frobenius algebras, then ϕ extends to an isomorphism $\psi : \mathcal{B}[W, G] \rightarrow \mathcal{B}[V, G]$.*

Proof. Given $\phi : \mathcal{Q}_W \rightarrow \mathcal{Q}_V$ an equivariant isomorphism of graded Frobenius algebras, we can apply Lemma 3.5 to obtain $\phi_i : \mathcal{Q}_{W_i} \rightarrow \mathcal{Q}_{V_i}$ that are also equivariant isomorphisms of graded Frobenius algebras. We can then extend Corollary 3.4 by induction in the case that $W = W_1 + \cdots + W_n$ and $V = V_1 + \cdots + V_n$ are sums of admissible polynomials in distinct variables such that each W_i is singularity equivalent to V_i , and G_i is a group that preserves both W_i and V_i for each i such that each group element of G_i fixes either all or none of the variables of W_i and V_i . \square

Theorem 3.6 actually applies to a large class of isomorphisms. For example, any diagonal isomorphism is equivariant.

Definition 3.7. Suppose $\phi : B_1 \rightarrow B_2$ is an isomorphism of B-models. Say that B_1 has basis $\{a_1, \dots, a_n\}$ and B_2 has basis $\{b_1, \dots, b_n\}$. We say that ϕ is *diagonal* if we can write $\phi(a_i) = c_i b_i$ for $c_i \in \mathbb{C}$ nonzero (possibly after reordering the basis elements).

Theorem 3.8. *Any diagonal isomorphism of Landau-Ginzburg B-models is equivariant.*

Proof. Suppose $\phi : B_1 \rightarrow B_2$ is a diagonal isomorphism of B-models. Let B_1 have basis $\{a_1, \dots, a_n\}$ and B_2 have basis $\{b_1, \dots, b_n\}$. Write $\phi(a_i) = c_i b_i$ for $c_i \in \mathbb{C}$ nonzero (reordering if necessary). Now notice

the following. For any $g \in G$,

$$\begin{aligned}\phi(g \cdot a_i) &= \phi(\det(g)a_i \circ g) = \det(g)\phi(a_i \circ g) = \det(g)c_i(b_i \circ g). \\ g \cdot \phi(a_i) &= g \cdot c_i b_i = \det(g)c_i(b_i \circ g).\end{aligned}$$

This happens since $a_i \circ g$ is just a constant times a_i . Because $\phi(g \cdot a_i) = g \cdot \phi(a_i)$ for each i , we see that ϕ is equivariant. \square

4 Examples

In the following examples, we will demonstrate how we can apply these results.

Example 4.1 (see Theorems 6.3 and 6.6 of Cordner (2016)). We can compute for all $n \geq 2$,

$$\mathcal{B}[x^2 + y^{2n}, \{\mathbf{0}\}] \xrightarrow{\sim} \mathcal{B}[x^2 + xy^n + y^{2n}, \{\mathbf{0}\}] \xleftarrow{\sim} \mathcal{B}[x^2 + xy^n, \{\mathbf{0}\}]$$

Each of these unorbifolded B-models has basis $\text{span}_{\mathbb{C}}\{1, y, \dots, y^{2n-2}\}$. We can define a map $\phi_1 : \mathcal{B}[x^2 + y^{2n}, \{\mathbf{0}\}] \rightarrow \mathcal{B}[x^2 + xy^n + y^{2n}, \{\mathbf{0}\}]$ by $\phi_1(y^a) = c^a y^a$, where c is a complex number that satisfies $c^{2n-2} = \frac{3}{4}$. We can also define a map $\phi_2 : \mathcal{B}[x^2 + xy^n, \{\mathbf{0}\}] \rightarrow \mathcal{B}[x^2 + xy^n + y^{2n}, \{\mathbf{0}\}]$ by $\phi_2(y^a) = c^a y^a$, where c is a complex number that satisfies $c^{2n-2} = -3$.

Lemma 4.2. *The map ϕ_1 is an isomorphism of graded Frobenius algebras.*

Proof. Let $W_\alpha = x^2 + \alpha xy^n + y^{2n}$ for $\alpha \in \mathbb{C}$, and let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \{\mathbf{0}\}]$. So $\mathcal{B}_0 = \mathbb{C}[x, y]/(2x, 2ny^{2n-1}) = \text{span}_{\mathbb{C}}\{1, y, \dots, y^{2n-2}\}$, which has dimension $2n - 1$. We also see that $\mathcal{B}_\alpha = \mathbb{C}[x, y]/(2x + \alpha y^n, n\alpha xy^{n-1} + 2ny^{2n-1}) = \text{span}_{\mathbb{C}}\{1, y, \dots, y^{2n-2}\}$, which has dimension $2n - 1$. \mathcal{B}_α has further relations $x = -\frac{\alpha}{2}y^n$ and $xy^{n-1} = -\frac{2}{\alpha}y^{2n-1}$. In each case $\deg(y^a) = \frac{a}{2n}$. The only possible map that can work in this case is diagonal.

W_α is nondegenerate when $2x + \alpha y^n = 0$ and $n\alpha xy^{n-1} + 2ny^{2n-1} = 0$. Solving for x yields $x = -\frac{\alpha}{2}y^n$. Substituting gives $-\frac{n\alpha^2}{2}y^{2n-1} + 2ny^{2n-1} = 0$ which implies $y^{2n-1}(-\frac{n\alpha^2}{2} + 2n) = 0$. So either $y = 0$, or $-\frac{n\alpha^2}{2} + 2n = 0$. Thus $\alpha^2 = 4$, showing that W_α is degenerate only for $\alpha = \pm 2$.

Since we are working with unorbifolded B-models, the product structure is relatively simple. We have that $y^a \star y^b = y^{a+b}$ if $a + b \leq 2n - 2$, and is equal to 0 otherwise.

To understand the pairing structure, we'll now compute the Hessian of W_α . We first compute

$\frac{\partial^2 W_\alpha}{\partial x^2} = 2$, $\frac{\partial^2 W_\alpha}{\partial x \partial y} = \frac{\partial^2 W_\alpha}{\partial y \partial x} = n\alpha y^{n-1}$, and $\frac{\partial^2 W_\alpha}{\partial y^2} = n(n-1)\alpha xy^{n-2} + 2n(2n-1)y^{2n-2}$. So

$$\begin{aligned} \text{Hess}(W_\alpha) &= 2[n(n-1)\alpha xy^{n-2} + 2n(2n-1)y^{2n-2}] - (n\alpha y^{n-1})^2 \\ &= 2[(n^2 - n)\alpha(-\frac{\alpha}{2}y^n)y^{n-2} + (4n^2 - 2n)y^{2n-2}] - n^2\alpha^2 y^{2n-2} \text{ substituting for } x, \\ &= -\alpha^2(n^2 - n)y^{2n-2} + (8n^2 - 4n)y^{2n-2} - n^2\alpha^2 y^{2n-2} \\ &= [(-2\alpha^2 + 8)n^2 + (\alpha^2 - 4)n]y^{2n-2}. \end{aligned}$$

Plugging in 0 for α yields $\text{Hess}(W_0) = (8n^2 - 4n)y^{2n-2}$. On \mathcal{B}_0 and \mathcal{B}_α we obtain a nonzero value for the pairing $\langle y^a, y^b \rangle$ precisely when $a + b = 2n - 2$. For nonzero \mathcal{B}_0 pairings, we obtain

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1} (8n^2 - 4n)y^{2n-2} \Rightarrow \frac{2n-1}{8n^2 - 4n} = \langle y^a, y^b \rangle \Rightarrow \frac{1}{4n} = \langle y^a, y^b \rangle.$$

For nonzero \mathcal{B}_α pairings, we obtain

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1} [(-2\alpha^2 + 8) + (\alpha^2 - 4)n]y^{2n-2} \Rightarrow \langle y^a, y^b \rangle = \frac{2n-1}{(-2\alpha^2 + 8)n^2 + (\alpha^2 - 4)n}.$$

We'll now construct a map $\phi : \mathcal{B}_0 \rightarrow \mathcal{B}_\alpha$, defined by $\phi = \text{diag}[1, c, c^2, \dots, c^{2n-2}]$. We'll state what value c should be in just a moment. First we'll check that ϕ preserves the product structure.

$$\begin{aligned} \phi(y^a \star y^b) &= \phi(y^{a+b}) = c^{a+b} y^{a+b} \\ \phi(y^a \star y^b) &= \phi(y^a) \star \phi(y^b) = c^a y^a \star c^b y^b = c^{a+b} y^{a+b}. \end{aligned}$$

For ϕ to preserve pairings, we require (assuming $a + b = 2n - 2$):

$$\begin{aligned} \frac{1}{4n} = \langle y^a, y^b \rangle_{\mathcal{B}_0} &= \langle \phi(y^a), \phi(y^b) \rangle_{\mathcal{B}_\alpha} \\ &= \langle c^a y^a, c^b y^b \rangle_{\mathcal{B}_\alpha} = c^{2n-2} \langle y^a, y^b \rangle_{\mathcal{B}_\alpha} = c^{2n-2} \left(\frac{2n-1}{(-2\alpha^2 + 8)n^2 + (\alpha^2 - 4)n} \right). \end{aligned}$$

Therefore c is any complex number satisfying $c^{2n-2} = \frac{(-2\alpha^2 + 8)n^2 + (\alpha^2 - 4)n}{4n(2n-1)} = \frac{(-2\alpha^2 + 8)n + (\alpha^2 - 4)}{4(2n-1)} = \frac{-2(\alpha^2 - 4)n + (\alpha^2 - 4)}{4(2n-1)} = \frac{-(2n-1)(\alpha^2 - 4)}{4(2n-1)} = -\frac{\alpha^2 - 4}{4}$. This gives us an isomorphism of graded Frobenius algebras. We obtain ϕ_1 by setting $\alpha = 1$. \square

Lemma 4.3. *The map ϕ_2 is an isomorphism of graded Frobenius algebras.*

Proof. Let $W_\alpha = x^2 + xy^n + \alpha y^{2n}$ for $\alpha \in \mathbb{C}$, and let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \{0\}]$. Then $\mathcal{B}_0 = \mathbb{C}[x, y]/(2x + y^n, nxy^{n-1}) = \text{span}_{\mathbb{C}}\{1, y, \dots, y^{2n-2}\}$ with relation $x = -\frac{1}{2}y^n$. Also, $\mathcal{B}_\alpha = \mathbb{C}[x, y]/(2x + y^n, nxy^{n-1} + 2n\alpha y^{2n-1}) = \text{span}_{\mathbb{C}}\{1, y, \dots, y^{2n-2}\}$ with relations $x = -\frac{1}{2}y^n$ and $xy^{n-1} = -2\alpha y^{2n-1}$.

To find when W_α is nondegenerate, we solve the equations $2x + y^n = 0$, $nxy^{n-1} + 2n\alpha y^{2n-1} = 0$. We

see that $x = -\frac{1}{2}y^n$. Substituting yields $n(-\frac{1}{2}y^n)y^{n-1} + 2n\alpha y^{2n-1} = 0$, so $(-\frac{n}{2} + 2n\alpha)y^{2n-1} = 0$. Hence $-\frac{n}{2} + 2n\alpha = 0$, which yields $\alpha = \frac{1}{4}$. This is our only point of nondegeneracy.

The product structure behaves the same as the example in Lemma 4.2. So we proceed to compute $\frac{\partial^2 W_\alpha}{\partial x^2} = 2$, $\frac{\partial^2 W_\alpha}{\partial x \partial y} = \frac{\partial^2 W_\alpha}{\partial y \partial x} = ny^{n-1}$, and $\frac{\partial^2 W_\alpha}{\partial y^2} = n(n-1)xy^{n-2} + 2n(2n-1)\alpha y^{2n-2}$. Therefore

$$\begin{aligned} \text{Hess}(W_\alpha) &= 2[n(n-1)xy^{n-2} + 2n(2n-1)\alpha y^{2n-2}] - (ny^{n-1})^2 \\ &= 2[-\frac{1}{2}n(n-1)y^{2n-2} + 2n(2n-1)\alpha y^{2n-2}] - n^2 y^{2n-2} \\ &= [-n(n-1) + 4n(2n-1)\alpha - n^2]y^{2n-2} \\ &= [(8\alpha - 2)n^2 + (-4\alpha + 1)n]y^{2n-2}. \end{aligned}$$

For \mathcal{B}_0 pairings, we find that

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1} (-2n^2 + n)y^{2n-2} \Rightarrow \langle y^a, y^b \rangle = \frac{2n-1}{-2n^2+n} = -\frac{1}{n}.$$

For \mathcal{B}_α pairings, we find that

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1} ((8\alpha - 2)n^2 + (-4\alpha + 1)n)y^{2n-2} \Rightarrow \langle y^a, y^b \rangle = \frac{2n-1}{(8\alpha - 2)n^2 + (-4\alpha + 1)n}.$$

(Noting, of course, that we use $a + b = 2n - 2$). To define $\phi : \mathcal{B}_0 \rightarrow \mathcal{B}_\alpha$ that preserves the pairing structure, we'll need

$$\begin{aligned} -\frac{1}{n} &= \langle y^a, y^b \rangle_{\mathcal{B}_0} = \langle c^a y^a, c^b y^b \rangle_{\mathcal{B}_\alpha} \\ &= c^{2n-2} \left(\frac{2n-1}{(8\alpha - 2)n^2 + (-4\alpha + 1)n} \right) \Rightarrow c^{2n-2} = -\frac{(8\alpha - 2)n^2 + (-4\alpha + 1)n}{n(2n-1)}. \end{aligned}$$

Hence $c^{2n-2} = -\frac{(8\alpha-2)n+(-4\alpha+1)}{2n-1} = -\frac{(2n-1)(4\alpha-1)}{2n-1} = -4\alpha + 1$. The map $\phi = \text{diag}[1, c, c^2, \dots, c^{2n-2}]$, which we checked before, gives us an isomorphism of graded Frobenius algebras. We obtain ϕ_2 by setting $\alpha = 1$. \square

Since the maps ϕ_1, ϕ_2 are diagonal, they are equivariant. If n is odd, then $G = \langle (\frac{1}{2}, \frac{1}{2}) \rangle$ fixes each polynomial. By Theorem 3.2, for all odd $n > 2$ we obtain extension isomorphisms ψ_1 for ϕ_1 and ψ_2 for ϕ_2 .

$$\begin{array}{ccccc} \mathcal{B}[x^2 + y^{2n}, G] & \xrightarrow{\psi_1} & \mathcal{B}[x^2 + xy^n + y^{2n}, G] & \xleftarrow{\psi_2} & \mathcal{B}[x^2 + xy^n, G] \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{B}[x^2 + y^{2n}, \{\mathbf{0}\}] & \xrightarrow{\phi_1} & \mathcal{B}[x^2 + xy^n + y^{2n}, \{\mathbf{0}\}] & \xleftarrow{\phi_2} & \mathcal{B}[x^2 + xy^n, \{\mathbf{0}\}] \end{array}$$

Applying mirror symmetry to B-models built with invertible polynomials, we get the following mirror

diagram.

$$\begin{array}{ccc}
\mathcal{A}[x^2 + y^{2n}, \langle (\frac{1}{2}, 0), (0, \frac{1}{2n}) \rangle] & \xrightarrow{\sim} & \mathcal{A}[x^2 y + y^n, \langle (-\frac{1}{2n}, \frac{1}{n}) \rangle] \\
\parallel & & \parallel \\
\downarrow & & \downarrow \\
\mathcal{A}[x^2 + y^{2n}, \langle (\frac{1}{2}, \frac{1}{2n}) \rangle] & \xrightarrow{\sim} & \mathcal{A}[x^2 y + y^n, \langle (\frac{n-1}{2n}, \frac{1}{n}) \rangle]
\end{array}$$

Here the unorbifolded B-models in the previous diagram correspond to the top row of A-models in the above diagram. The orbifolded B-models of the previous diagram correspond to the A-models on the bottom row in the above diagram. Notice that the isomorphism $\mathcal{A}[x^2 + y^{2n}, \langle (\frac{1}{2}, \frac{1}{2n}) \rangle] \cong \mathcal{A}[x^2 y + y^n, \langle (\frac{n-1}{2n}, \frac{1}{n}) \rangle]$ is the result of the B-model isomorphisms we just computed together with mirror symmetry. Further note that the groups used for these A-models are distinct. Therefore, this is a new isomorphism of A-models that does not stem from the Group-Weights theorem. Hence Theorem 3.2 tells us not only about isomorphisms of B-models, but can also be used to find new isomorphisms between A-models.

Example 4.4. In singularity theory, it is known that adding *quadratic forms* in distinct variables to polynomials will do nothing to affect the type of singularity defined. Such stabilization of singularities and their related Landau-Ginzburg A-model structure has been studied by Francis (2012). For Landau-Ginzburg B-models, it is not immediately clear how the orbifolded Milnor ring structure of such an augmented polynomial will be affected.

Building off of Example 4.1, for each odd integer $n > 2$ let $W_n^{(1)} = x^2 + y^{2n}$, $W_n^{(2)} = x^2 + xy^n + y^{2n}$, and $W_n^{(3)} = x^2 + xy^n$. Consider also the polynomial $V = z^2 + w^2$. We already know that $\mathcal{Q}_{W_n^{(1)}}$, $\mathcal{Q}_{W_n^{(2)}}$, and $\mathcal{Q}_{W_n^{(3)}}$ are isomorphic to each other under equivariant maps. We also immediately see that \mathcal{Q}_V is isomorphic to \mathcal{Q}_V under the identity map, which is equivariant.

The group $G = \langle (\frac{1}{2}, \frac{1}{2}) \rangle$ preserves each of $W_n^{(1)}$, $W_n^{(2)}$, $W_n^{(3)}$, and V . Note also that the pair (V, G) is well behaved. Let $G_1 = \{(0, 0, 0, 0)\}$, $G_2 = G \times \{(0, 0)\} = \langle (\frac{1}{2}, \frac{1}{2}, 0, 0) \rangle$, $G_3 = \{(0, 0)\} \times G = \langle (0, 0, \frac{1}{2}, \frac{1}{2}) \rangle$, and $G_4 = G \times G = \langle (\frac{1}{2}, \frac{1}{2}, 0, 0), (0, 0, \frac{1}{2}, \frac{1}{2}) \rangle$. By Corollary 3.4, we have following B-model isomorphisms:

$$\begin{aligned}
\mathcal{B}[W_n^{(1)} + V, G_1] &\cong \mathcal{B}[W_n^{(2)} + V, G_1] \cong \mathcal{B}[W_n^{(3)} + V, G_1], \\
\mathcal{B}[W_n^{(1)} + V, G_2] &\cong \mathcal{B}[W_n^{(2)} + V, G_2] \cong \mathcal{B}[W_n^{(3)} + V, G_2], \\
\mathcal{B}[W_n^{(1)} + V, G_3] &\cong \mathcal{B}[W_n^{(2)} + V, G_3] \cong \mathcal{B}[W_n^{(3)} + V, G_3], \\
\mathcal{B}[W_n^{(1)} + V, G_4] &\cong \mathcal{B}[W_n^{(2)} + V, G_4] \cong \mathcal{B}[W_n^{(3)} + V, G_4].
\end{aligned}$$

We see that in these cases, equivalent singularities still yield the same orbifolded Milnor ring structure. We can take this one step further. If (W, G) and (V, G) are any well-behaved pairs where W is singularity equivalent to V , and (U, H) is a well-behaved pair where U is a quadratic form in distinct variables from W , V and H is some orbifold group for U , we can apply Corollary 3.4 to find that $\mathcal{B}[W + U, G \times H] \cong$

$\mathcal{B}[V + U, G \times H]$.

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