Erasures vs. Errors in Local Decoding and Property Testing

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Goals

Study the role of **erasures in local decoding**

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Use our understanding to **separate erasures and errors in property testing**

Overview of Results

- **Local list decoding in the presence of erasures**
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  - Constant vs. $\Omega(\log n)$ separation between erasure-resilient testing and tolerant testing

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- **Approximate local list decoding in the presence of erasures**
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- **Relationship between local decoding in the presence of erasures and in the presence of errors**

Locally (Unique or List) Decodable Codes

- **Locally decodable codes** [Babai Fortnow Levin Szegedy 91, Gemmel Lipton Rubinfeld Sudan Wigderson 91, Gemmel Sudan 92, Blum Luby Rubinfeld 93, Polishchuk Spielman 94, Beimel Ishai Kushilevitz Raymond 02, Yekhanin 08, Ben-Aroya Efremenko TaShma 10, Dvir Gopalan Yekhanin 11, Efremenko 12, …]
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  - Given oracle access to codeword, with high probability, obtain a list of descriptions of local decoders of each candidate message
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Can locally decodable codes perform better with erasures than with errors?

Local List Decoding of Hadamard Code

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Type of corruptions	Corruption tolerance α	List size, ℓ	Number of queries, q	Upper bound	Lower bound
Errors	$\alpha \in \left(0, \frac{1}{2}\right)$	$\Theta\left(\frac{1}{\left(\frac{1}{2} - \alpha\right)^2}\right)$	$\Theta\left(\frac{1}{\left(\frac{1}{2} - \alpha\right)^2}\right)$	[Goldreich Levin 89]	[Blinovsky 86, Guruswami Vadhan 10, Grinberg Shaltiel Viola 18]

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Erasures	$\alpha \in (0,1)$	$O\left(\frac{1}{1-\alpha}\right)$	$\Theta\left(\frac{1}{1-\alpha}\right)$	**new**	Implicit in [Grinberg Shaltiel Viola 18]

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- Hadamard: $\{0,1\}^k \to \{0,1\}^{2^k}$; $\text{Hadamard}(x) = (\langle x, y\rangle)_{y\in\{0,1\}^k}$
- Impossible to decode when fraction of errors is $\geq 1/2$.

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Erasure-decoding better than error-decoding

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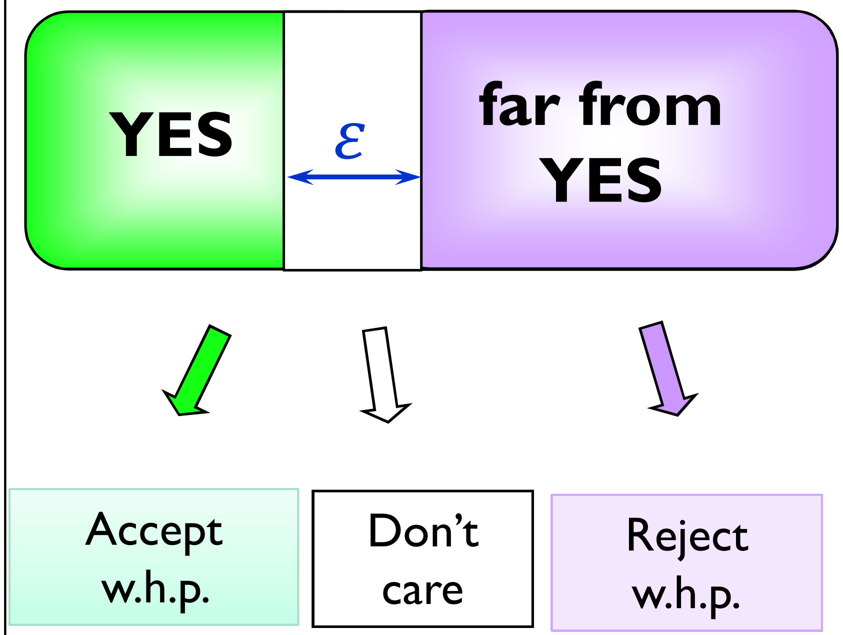
Property Testing

Property Tester

[Rubinfeld Sudan 96,
Goldreich Goldwasser Ron 98]



Property = Set of all YES instances

1	1	3	3	5	5	7	7	9	9

sorted array

2	1	4	3	6	5	8	7	9	0

1/2-far from sorted

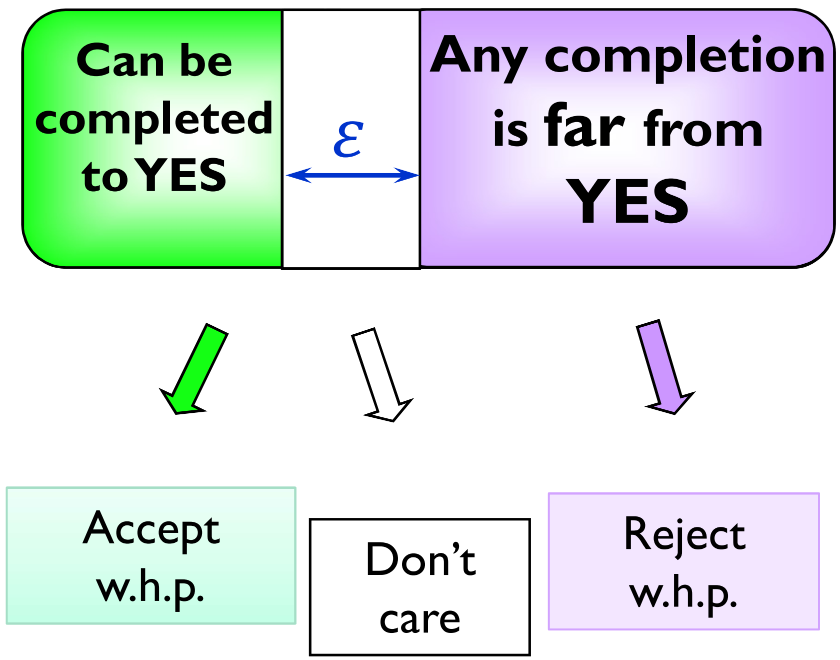
Two objects are at distance ε = they differ in an ε fraction of places

Erasure-Resilient and Tolerant Property Testing

Erasure-Resilient Property Tester

[Dixit Raskhodnikova Thakurta Varma 16]

$\leq \alpha$ fraction of the input is erased adversarially



α-erasure-resilient ε-testing

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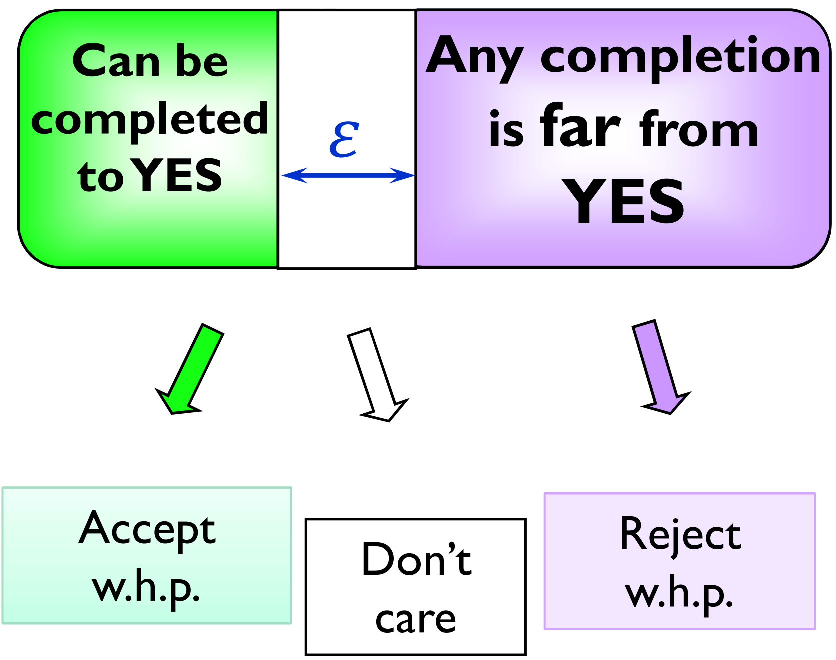
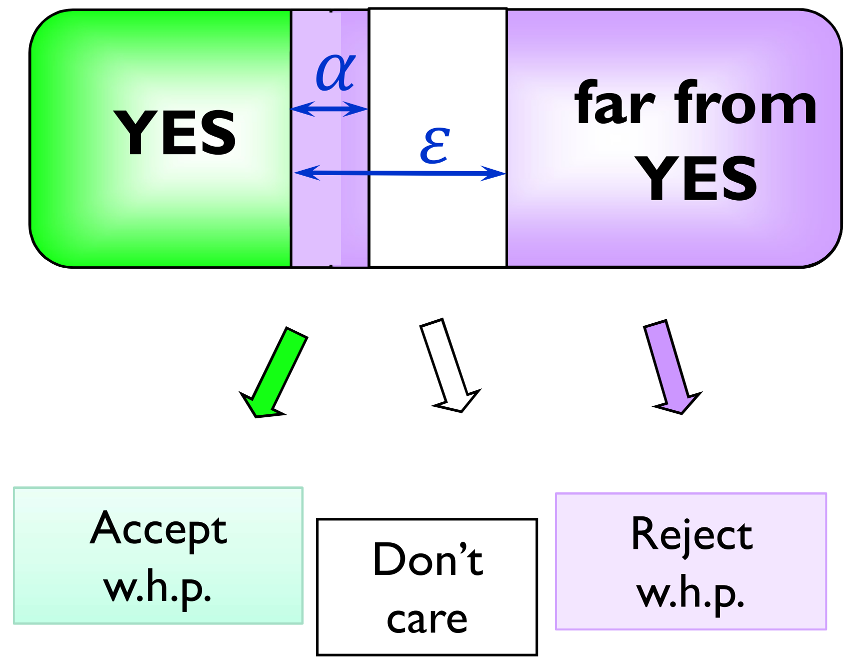


Tolerant Property Tester

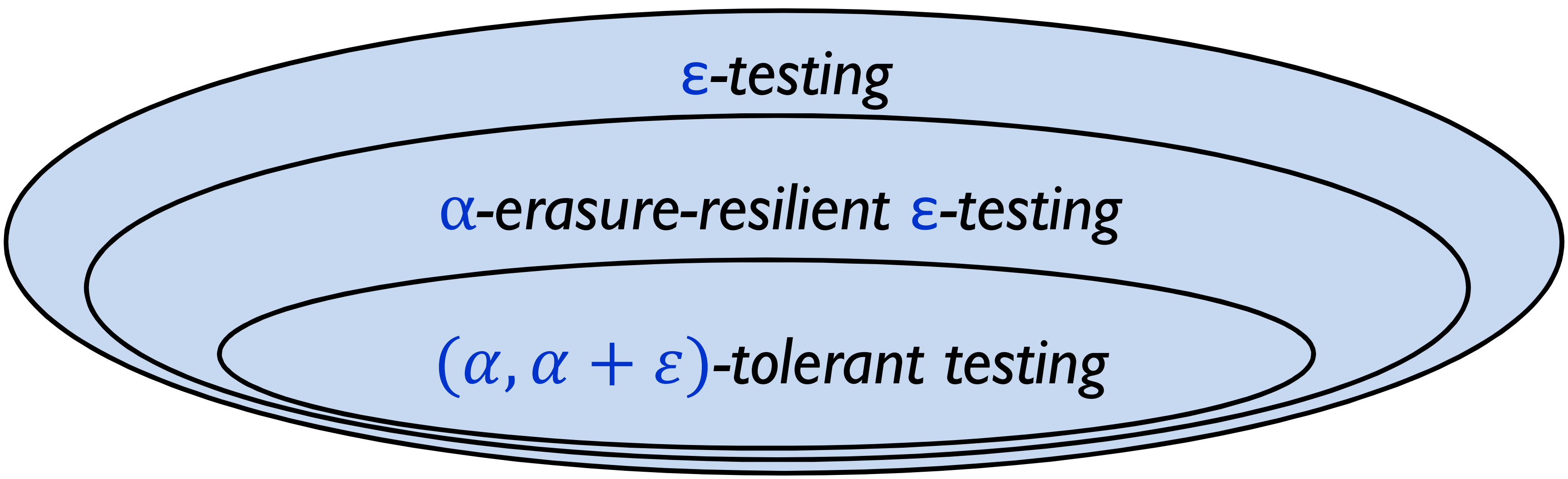
[Parnas Ron Rubinfeld 06]

$\leq \alpha$ fraction of the input is erroneous



α-erasure-resilient ε-testing **(α, ε)-tolerant testing**

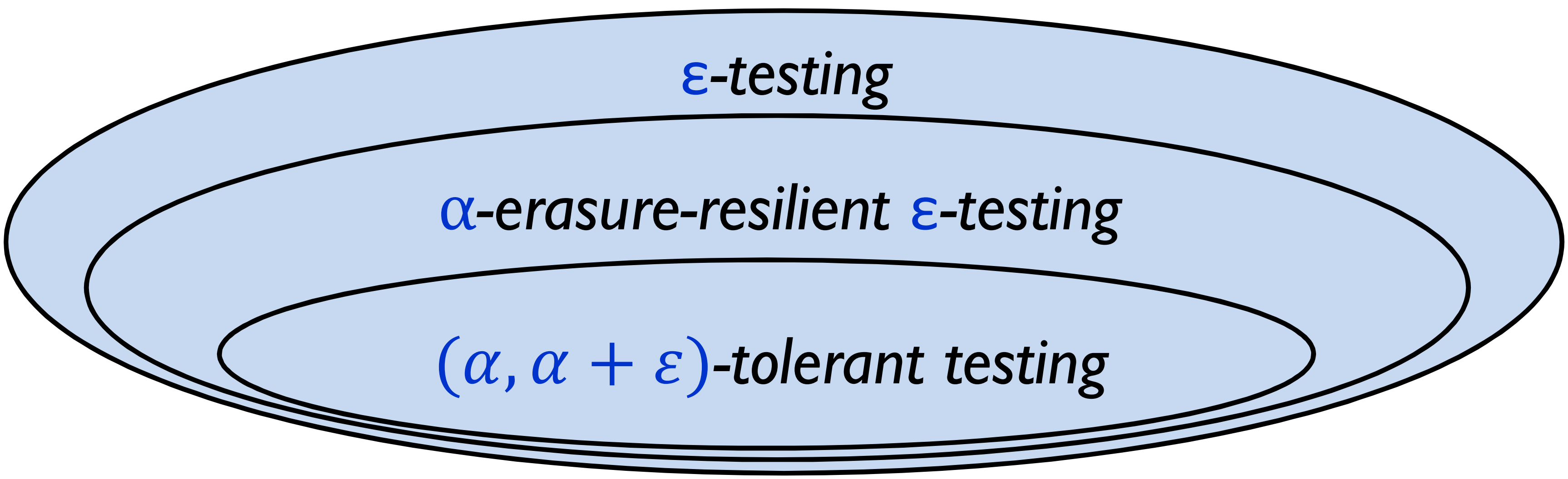
Relationships Between Models



Relationships Between Models

Containments are strict:

- [Fischer Fortnow 05]: standard vs. tolerant
- [Dixit Raskhodnikova Thakurta Varma 16]: standard vs. erasure-resilient
- **Our Result**: erasure-resilient vs. tolerant



Our First Separation Result

First Separation Theorem

There is a property of $\boldsymbol{n}$-bit strings that

- can be erasure-resiliently tested with **constant** query complexity,
- but requires $\boldsymbol{\Omega(\log n)}$ queries for tolerant testing.

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Main Tools

Separation between erasures and errors in local list decoding Hadamard codes

PCPs of proximity [BenSasson Goldreich Harsha Sudan Vadhan 06, Dinur Reingold 06, Dinur 07]

(~ PCPs for property testing problems)

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Error-tolerant testing is harder than erasure-resilient testing in general.

Strengthened Separation Result

Strengthened Separation Theorem

There is a property of $\boldsymbol{n}$-bit strings that

- can be erasure-resiliently tested with **constant** query complexity,
- but requires ~~Ω(log n)~~ $\boldsymbol{n}^{\boldsymbol{\Omega(1)}}$ queries for tolerant testing.

Main Tools

Separation between erasures and errors in **approximate** local list decoding ~~Hadamard codes~~

PCPs of proximity [BenSasson Goldreich Harsha Sudan Vadhan 06, Dinur Reingold 06, Dinur 07]

(~ PCPs for property testing problems)

Error-tolerant testing is **much** harder than erasure-resilient testing in general.

Errors and Erasures in Local Decoding

- **Local decoding implies local erasure-decoding**
  - locally decodable from at most an α fraction of errors $\Rightarrow$ locally decodable from at most an 2α fraction of erasures
  - Also holds for local list decoding and approximate local list decoding

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- **Local decoding implies local erasure-decoding**
  - locally decodable from at most an α fraction of errors $\Rightarrow$ locally decodable from at most an 2α fraction of erasures
  - Also holds for local list decoding and approximate local list decoding
- **Local erasure-decoding implies local decoding** (up to some parameters)
  - locally decodable from at most an α fraction of erasures using q queries $\Rightarrow$ locally decodable from at most an $\alpha/O(q^2 \cdot 9^q)$ fraction of errors using $O(q \cdot 3^q)$ queries

Open Questions

- Even stronger separation between erasure-resilient and tolerant testing -- constant vs. linear?
- Separation between errors and erasures for a "natural" property?
- Constant-query, constant list size, local list erasure-decodable codes with inverse polynomial rate?

Thank you!