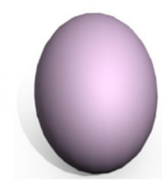
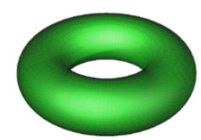
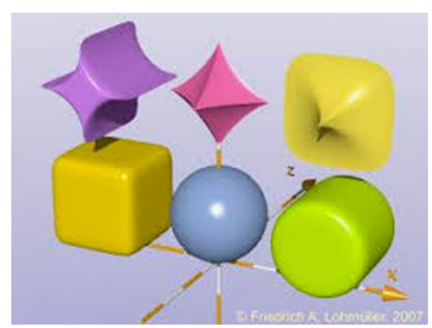
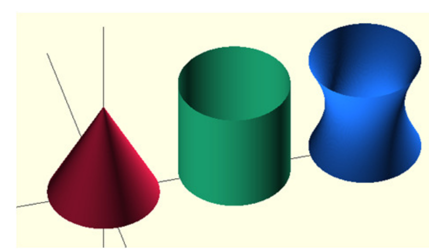
3D Geometry Representation for Computer Graphics

November 3, 2016

Outline

- Implicit *vs*. parametric equations for surfaces
- Surface normals
- Surface types discussed in this lecture
  - Plane
  - Quadrics: Sphere, ellipsoid, cylinder, cone, etc.
  - Toroid
  - Superquadric
  - Supertoroid

Sphere

Implicit equation for sphere centered at origin:

$$f(x, y, z) = x^2 + y^2 + z^2 - r^2$$

3D point *(x,y,z)* is on, outside, or inside sphere:

$$f(x, y, z) = 0 \qquad f(x, y, z) > 0 \qquad f(x, y, z) < 0$$

Parametric equation for sphere:

$$x = r\cos\varphi\cos\theta \qquad \frac{-\pi}{2} \le \phi \le \frac{\pi}{2}$$

$$y = r\cos\varphi\sin\theta$$

$$z = r\sin\varphi \qquad -\pi \le \theta \le \pi$$

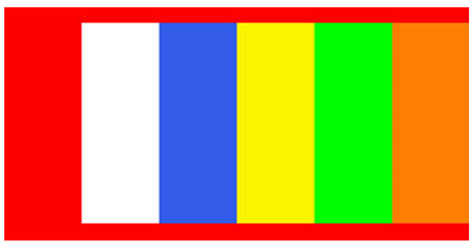
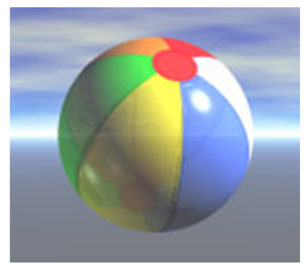
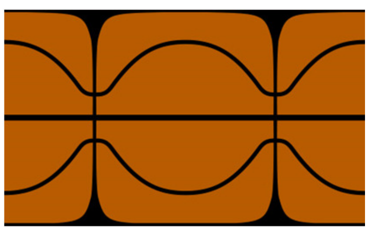
Sphere

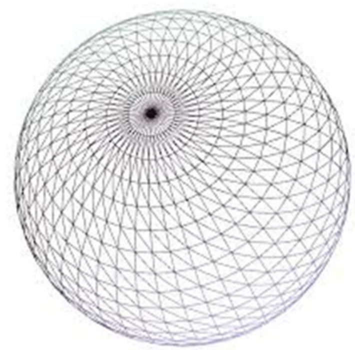
Parametric equation, rewritten in vector form:

$$\mathbf{p}(u,v) = \begin{bmatrix} r\cos u \cos v \\ r\cos u \sin v \\ r\sin u \\ 1 \end{bmatrix} \qquad \begin{matrix} \frac{-\pi}{2} \le u \le \frac{\pi}{2} \\ -\pi \le v \le \pi \end{matrix}$$

Parametric Surface Equations

- Parametric form is good for generating triangle mesh
- The *(u,v)* parameters of each vertex can be used for applying a 2D texture map to the surface



Texture maps from:
http://www.robinwood.com/Catalog/FreeStuff/Textures/TexturePages/BallMaps.html

Ellipsoid

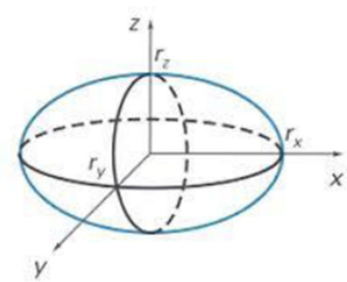
Implicit equation for ellipsoid centered at origin:

$$f(x,y,z) = \left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 - 1$$

Parametric equation for ellipsoid:

$$\mathbf{p}(u,v) = \begin{bmatrix} r_x \cos u \cos v \\ r_y \cos u \sin v \\ r_z \sin u \\ 1 \end{bmatrix}$$

$$\frac{-\pi}{2} \le u \le \frac{\pi}{2}$$
$$-\pi \le v \le \pi$$



Implicit Equation: Quadric Form

Implicit equation for general quadric surface can be written in terms of symmetric matrix **Q**:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix}$$

Equivalent to:

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

Sphere

Implicit equation:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -r^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Equivalent to:

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

$$f(x,y,z) = x^2 + y^2 + z^2 - r^2$$

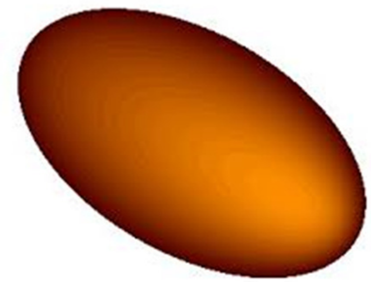
$$f(x,y,z) = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} - 1$$

Ellipsoid

Implicit equation:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0$$

$$\mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} = \begin{bmatrix} \frac{1}{r_x^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r_y^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r_z^2} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



Equivalent to:

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

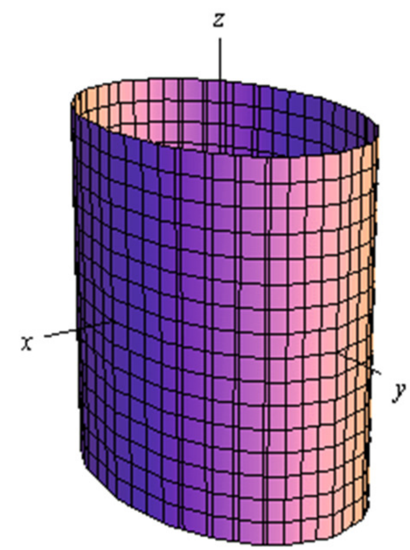
$$f(x,y,z) = \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} - 1$$

Cylinder Implicit Equation

Infinite cylinder (along z axis), elliptical cross-section:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0$$

$$\mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} = \begin{bmatrix} \frac{1}{r_x^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r_y^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



Equivalent to:

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

$$f(x,y,z) = \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} - 1$$

Parametric Surface Equations

Ellipsoid:

$$\mathbf{p}(u,v)=\begin{bmatrix} r_x \cos u \cos v \\ r_y \cos u \sin v \\ r_z \sin u \\ 1 \end{bmatrix} \qquad \frac{-\pi}{2} \le u \le \frac{\pi}{2}, \quad -\pi \le v \le \pi$$

Cylinder with elliptical cross-section:

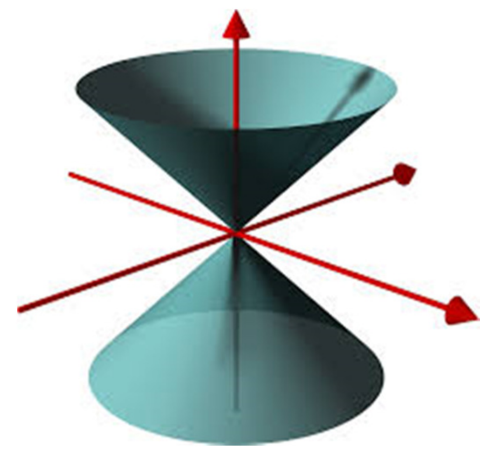
$$\mathbf{p}(u,v)=\begin{bmatrix} r_x \cos v \\ r_y \sin v \\ u \\ 1 \end{bmatrix} \qquad u_{min} \le u \le u_{max}, \quad -\pi \le v \le \pi$$

Cone Implicit Equation

Infinite cone (along z axis), elliptical cross-section:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0$$

$$\mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} = \begin{bmatrix} \frac{1}{r_x^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r_y^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{s^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Equivalent to:

$$f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

$$f(x,y,z) = \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} - \frac{z^2}{s^2}$$

Parametric Surface Equation

Cone with elliptical cross-section:

$$\mathbf{p}(u,v) = \begin{bmatrix} ur_x \cos v \\ ur_y \sin v \\ u \\ 1 \end{bmatrix} \qquad \begin{matrix} u_{min} \le u \le u_{max} \\ -\pi \le v \le \pi \end{matrix}$$

Plane Implicit Equation

Given plane equation $ax + by + cz - d = 0$

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0$$

$$\mathbf{Q} = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{a}{2} \\ 0 & 0 & 0 & \frac{b}{2} \\ 0 & 0 & 0 & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} & -d \end{bmatrix}$$

Equivalent to:

$$f(x, y, z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J = 0$$

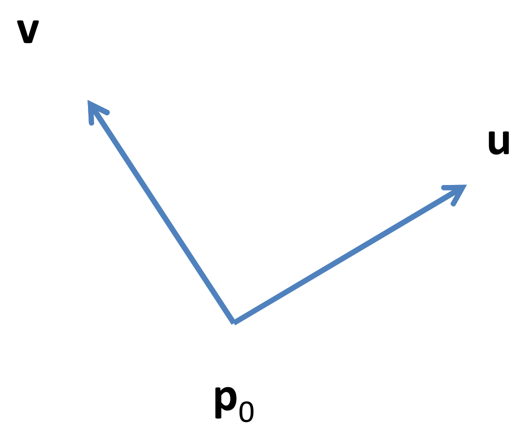
$$f(x, y, z) = ax + by + cz - d$$

Parametric Plane Equation

There are a number of ways to parameterize a plane surface. One is to use basis vectors **u**, **v** and origin $\mathbf{p}_0$

$$\mathbf{p}(u,v) = \mathbf{p}_0 + u\mathbf{u} + v\mathbf{v}$$

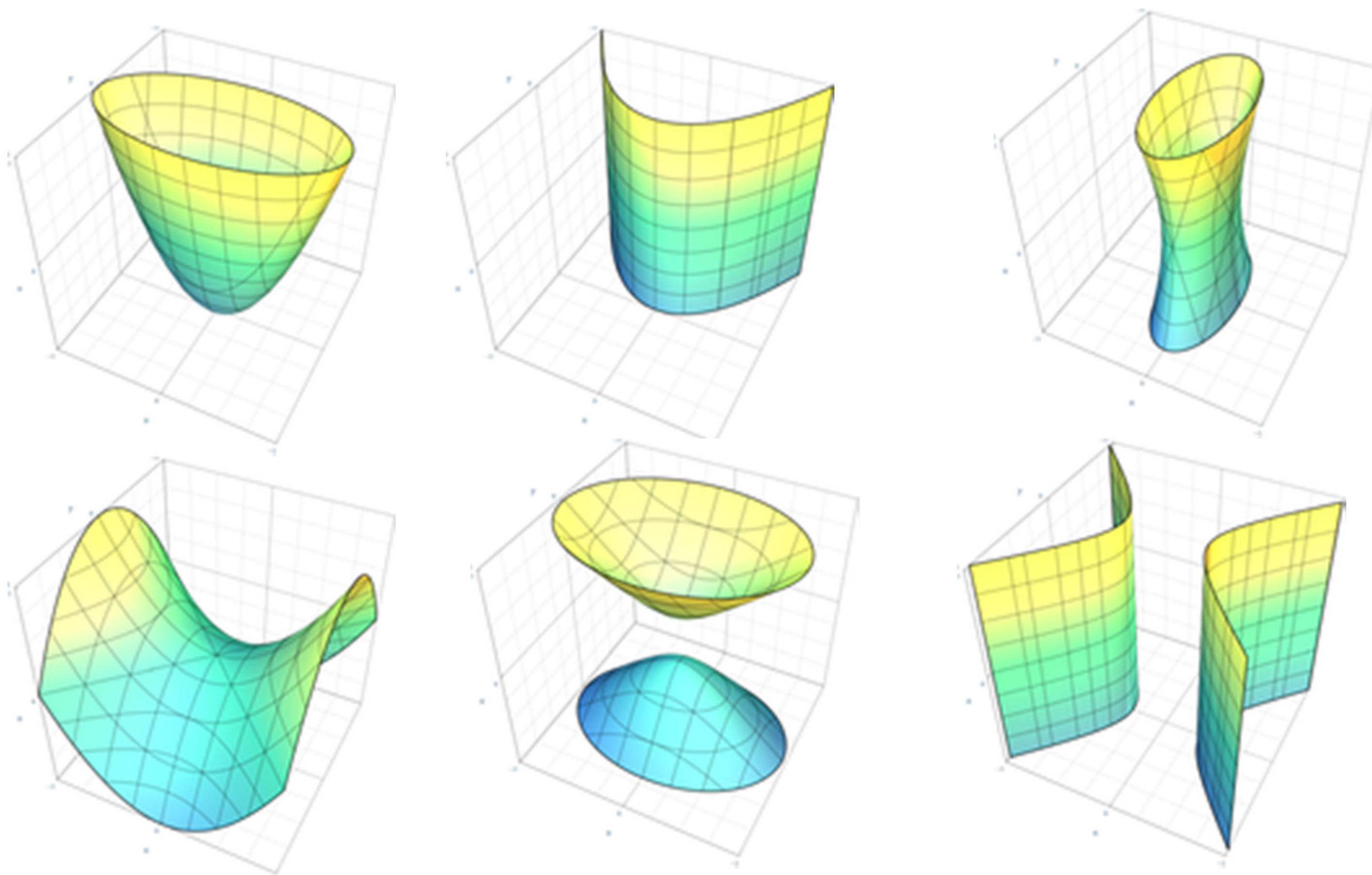
$$\mathbf{p}(u,v) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} + u \begin{bmatrix} x_u \\ y_u \\ z_u \\ 0 \end{bmatrix} + v \begin{bmatrix} x_v \\ y_v \\ z_v \\ 0 \end{bmatrix}$$



Basis vectors **u v** are orthogonal in this case.

Quadric Surfaces

See wikipedia page for more examples

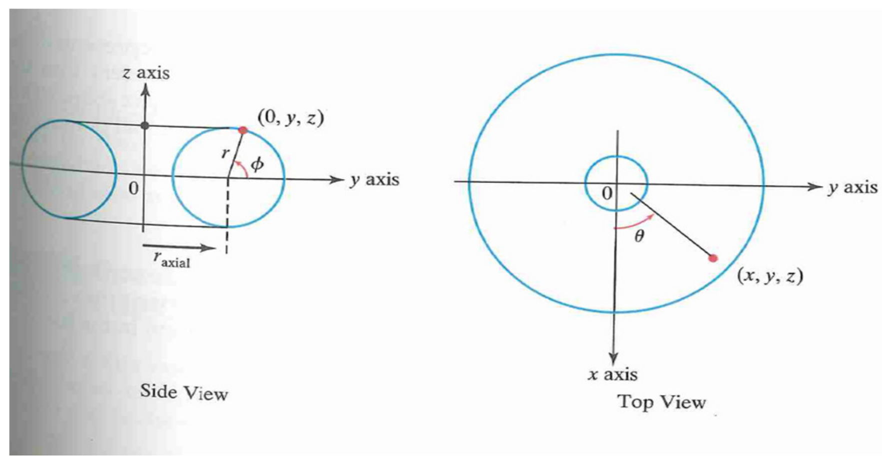


Torus

Implicit equation for torus with circular cross-section centered at origin:

$$f(x,y,z)=\left(\sqrt{x^2+y^2}-r_{axial}\right)+z^2-r^2$$

Parametric equation:

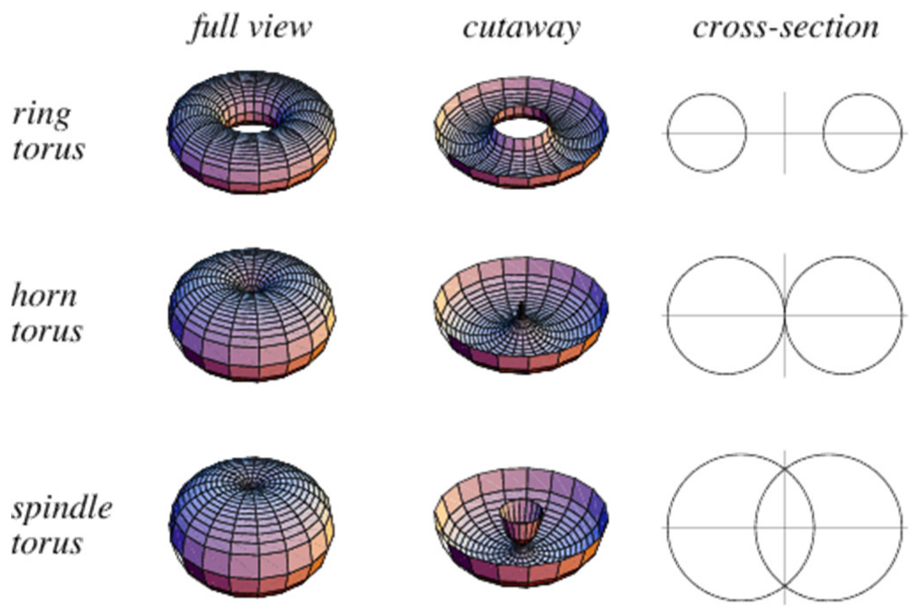


$$p(u,v)=\begin{bmatrix}(r_{axial}+r\cos u)\cos v\\(r_{axial}+r\cos u)\sin v\\r\sin u\\1\end{bmatrix}$$

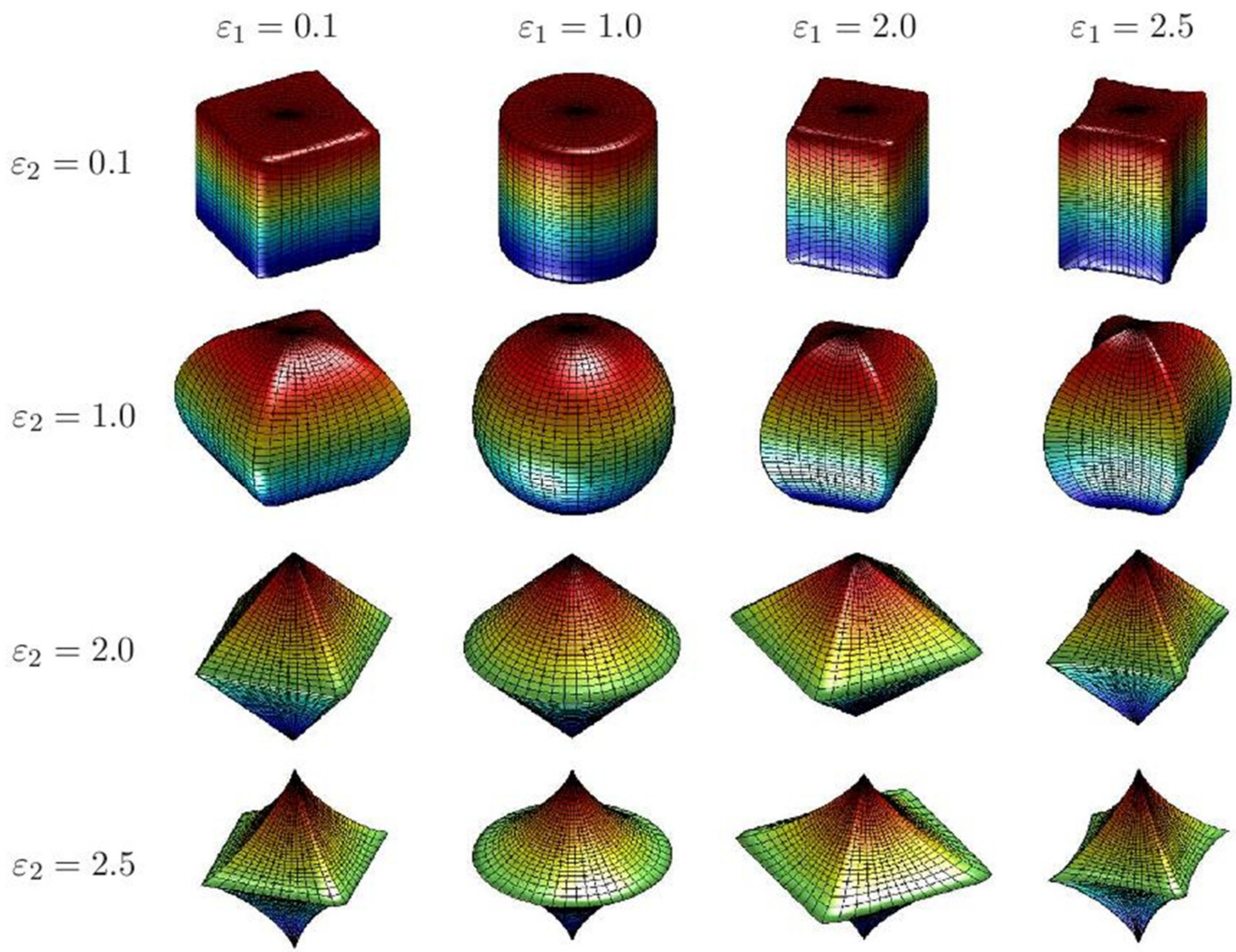
$$-\pi\le u\le\pi$$

$$-\pi\le v\le\pi$$

Varying the Radius: r_{axial}



Superquadric Ellipsoids



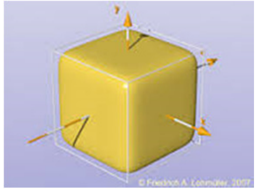
Superquadric Ellipsoids

Implicit equation:

$$f(x,y,z)=\left[\left(\frac{x}{r_x}\right)^{\frac{2}{\varepsilon_2}}+\left(\frac{y}{r_y}\right)^{\frac{2}{\varepsilon_2}}\right]^{\frac{\varepsilon_2}{\varepsilon_1}}+\left(\frac{z}{r_z}\right)^{\frac{2}{\varepsilon_1}}-1$$

Parametric equation:

$$p(u,v)=\begin{bmatrix} r_x\,\mathrm{sign}(\cos u)\left|\cos u\right|^{\varepsilon_1}\mathrm{sign}(\cos v)\left|\cos v\right|^{\varepsilon_2} \\ r_y\,\mathrm{sign}(\cos u)\left|\cos u\right|^{\varepsilon_1}\mathrm{sign}(\sin v)\left|\sin v\right|^{\varepsilon_2} \\ r_z\,\mathrm{sign}(\sin u)\left|\sin u\right|^{\varepsilon_1} \\ 1 \end{bmatrix}$$



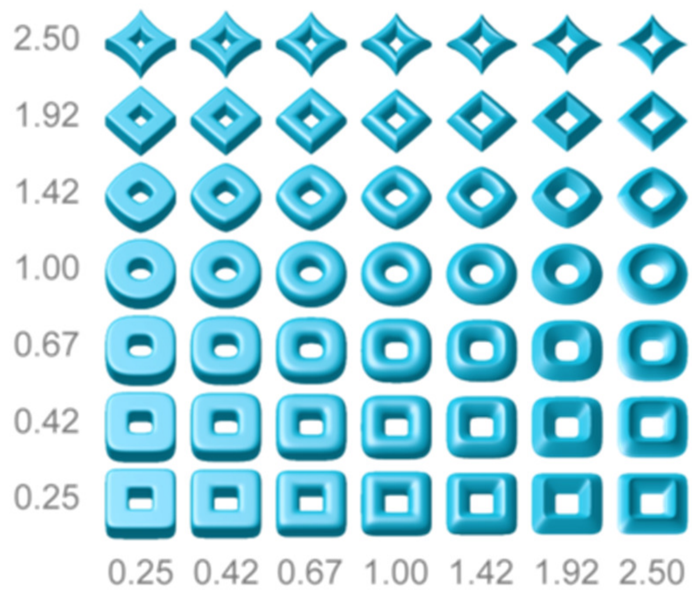
$$\frac{-\pi}{2}\le u\le\frac{\pi}{2},-\pi\le v\le\pi$$

$\varepsilon_1>0,\varepsilon_2>0$

$\varepsilon_1=\varepsilon_2=1$ is ellipsoid

$\varepsilon_1=\varepsilon_2=0.1$ is box-like shape

Supertoroids!



Supertoroid

Implicit equation:

$$f(x,y,z) = \left[\left[\left(\frac{x}{r_x}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{r_y}\right)^{\frac{2}{\varepsilon_2}} \right]^{\frac{\varepsilon_2}{2}} - r_{axial} \right]^{\frac{2}{\varepsilon_1}} + \left(\frac{z}{r_z}\right)^{\frac{2}{\varepsilon_1}} - 1$$

Parametric equation:

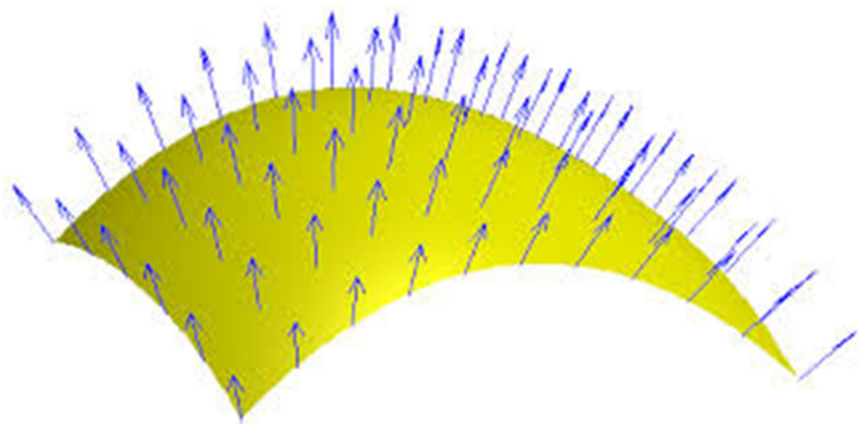
$$p(u,v) = \begin{bmatrix} r_x \operatorname{sign}(\cos u)\left|\cos u\right|^{\varepsilon_1} \left(r_{axial} + \operatorname{sign}(\cos v)\left|\cos v\right|^{\varepsilon_2}\right) \\ r_y \operatorname{sign}(\cos u)\left|\cos u\right|^{\varepsilon_1} \left(r_{axial} + \operatorname{sign}(\sin v)\left|\sin v\right|^{\varepsilon_2}\right) \\ r_z \operatorname{sign}(\sin u)\left|\sin u\right|^{\varepsilon_1} \\ 1 \end{bmatrix}$$

$$-\pi \le u \le \pi, -\pi \le v \le \pi$$

Computing Surface Normals

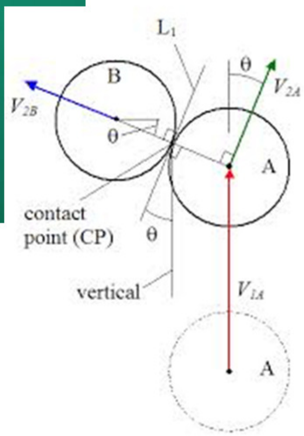
Surface Normal

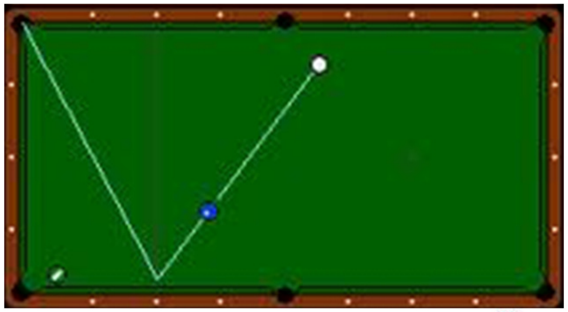
- Surface normal at each vertex can be used in illumination and shading of the mesh.

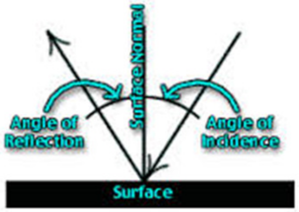


Surface Normal

- Advanced topic: Surface normals are also used in game physics and animation: computing collision response, friction, etc., where surfaces are in contact.



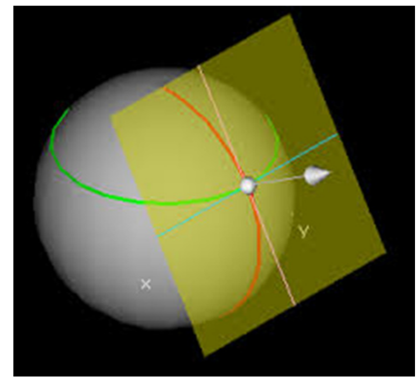




Computing the Surface Normal

- Parametric form:
  - Take partial derivative (tangent to surface) along each parametric direction.
  - Cross-product yields the normal.

$$\mathbf{n}(u,v) = \frac{\partial}{\partial u}\mathbf{p}(u,v) \otimes \frac{\partial}{\partial v}\mathbf{p}(u,v)$$



- Use 3D (non-homogeneous form), for instance sphere:

$$\mathbf{p}(u,v) = \begin{bmatrix} r\cos u \cos v \\ r\cos u \sin v \\ r \sin u \end{bmatrix}$$

Computing Sphere Surface Normal

$$\mathbf{n}(u,v) = \frac{\partial}{\partial u}\begin{bmatrix} r\cos u\cos v \\ r\cos u\sin v \\ r\sin u \end{bmatrix} \otimes \frac{\partial}{\partial v}\begin{bmatrix} r\cos u\cos v \\ r\cos u\sin v \\ r\sin u \end{bmatrix}$$

$$\mathbf{n}(u,v) = \begin{bmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{bmatrix} \otimes \begin{bmatrix} -r\cos u\sin v \\ r\cos u\cos v \\ 0 \end{bmatrix}$$

$$\frac{\mathbf{n}(u,v)}{\|\mathbf{n}(u,v)\|} = \begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{bmatrix}$$

Computing Surface Normal from Implicit Equation

- Given the implicit equation for a surface, we can obtain the normal at (x,y,z) by taking the gradient:

$$\mathbf{n}(x,y,z) = \nabla f(x,y,z)$$

$$\mathbf{n}(x,y,z) = \begin{bmatrix} \dfrac{\partial}{\partial x} f(x,y,z) \\ \dfrac{\partial}{\partial y} f(x,y,z) \\ \dfrac{\partial}{\partial z} f(x,y,z) \end{bmatrix}$$

Computing Sphere Surface Normal from Implicit Equation

Given a point on surface $f(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$

$$\mathbf{n}(x, y, z) = \nabla f(x, y, z)$$

$$\mathbf{n}(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2 - r^2\right) \\ \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2 - r^2\right) \\ \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2 - r^2\right) \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

Quadric Surface Normal from Implicit Equation

Given a point $\mathbf{x}$ on surface $\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0$

$$\mathbf{n}(\mathbf{x}) = \nabla\left(\mathbf{x}^T\mathbf{Q}\mathbf{x}\right)$$

$$\mathbf{n}(\mathbf{x}) = 2\mathbf{Q}\mathbf{x}$$