Randomness in Computing

CS 237

LECTURE 1

Randomness in Computing

- Course information
- Uses of probability in CS
- Verifying polynomial identities

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Course information

1. Course staff
2. Course website(s)
3. Piazza bonus
4. Prerequisites
5. Textbook(s)
6. TopHat
7. Syllabus
8. Homework logistics
9. Collaboration policy
10. Exams and grading

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Tips for the course

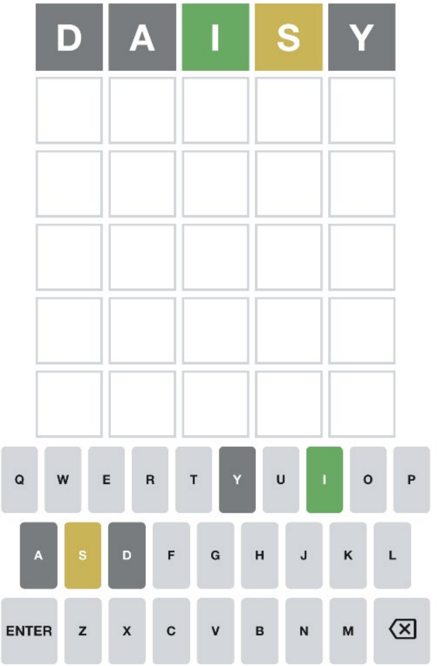
- Concepts in this course take some time to sink in: be careful not to fall behind.
- Prepare for each lecture by reviewing material from the previous lecture and doing assigned reading.

- Attend the *lectures*: some material will be presented on the blackboard (and some of it is not in the book).
- Attend the *discussions*: practice problem solving.
- Take advantage of *office hours*.

- Be active in lectures, discussions, and on piazza.
- Study with a friend: do exercises and quiz each other.
- Allocate lots of time for the course: comparable to a project course, but spread more evenly.

Tips for the course: HW

- Start working on HW early.
- Spread your HW time over multiple days.
- You can work in groups (up to 4 people), but spend 1-2 hours thinking about it on your own before your group meeting.



Tips: learning problem solving

To learn problem solving, you have to do it:

- Try to think how you would solve any presented problem before you read/hear the answer.
- Do exercises in addition to HW
  - do solved exercises in the supplementary textbook

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Tips: how to read a math text

- Not like reading a mystery novel.
- The goal is not to get the answers, but to learn the techniques.
- Always try to foresee what is coming next.
- Always think how you would approach a problem before reading the solution.
- This applies to things that are not explicitly labeled as problems.

Picture source: pngtree.com

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Skills we will work on

- Mathematical reasoning
- Expressing your ideas
  - abstractly (suppress inessential details)
  - precisely (rigorously)
- Probabilistic thinking
- A bit of algorithmic thinking
- Problem solving
- Computer simulations of probabilistic experiments
- Having **FUN** with all of the above!!!

Could they ask me questions

about CS 237 material on job interviews?

- You bet.

Uses of Probability in Computing

- To speed up algorithms.
- To enable new applications:
  - Symmetry breaking in distributed algorithms, cryptography, privacy, online games and gambling.
- To simulate real world events in physical systems: model them as happening randomly.
- To analyze algorithms when data is generated from some distribution:
  - learning theory, data compression.
- To analyze algorithms when errors happen randomly
  - error-correcting codes.
- Analyzing statistics from sampling.

Probability in CS Curriculum

- CS 350: Fundamentals of Computing Systems
- CS 507: Introduction to Optimization in Computing and Machine Learning
- CS 542: Machine Learning
- CS 535: Complexity Theory
- CS 537: Randomness in Computing
- CS 558: Introduction to Network Security
- …

Verifying Polynomial Identities

- $(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \stackrel{?}{\equiv} x^6 - 7x + 37$

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Polynomials

A polynomial in variable x is a function of the form

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0.$$

degree $d \geq 0$ and integer

variable

coefficients

Polynomial	Degree	Example	General form
Constant	0	3	$p(x) = \boldsymbol{a}$
Linear	1	$-7x - 2$	$p(x) = \boldsymbol{a}x + \boldsymbol{b}$
Quadratic	2	$x^2 - 4x + 3$	$p(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$
Cubic	3	$x^3 - 1$	$p(x) = \boldsymbol{a}x^3 + \boldsymbol{b}x^2 + \boldsymbol{c}x + \boldsymbol{d}$

Roots of a Polynomial

A polynomial in variable x is a function of the form

degree

$$p(x) = a_d x^{d} + a_{d-1} x^{d-1} + \cdots + a_0 \, .$$

- Number r is a **root** of $p(x)$ if $p(r) = 0$.

  Ex. $p(x) = x^2 - 9$ has two roots

  namely, 3 and -3.

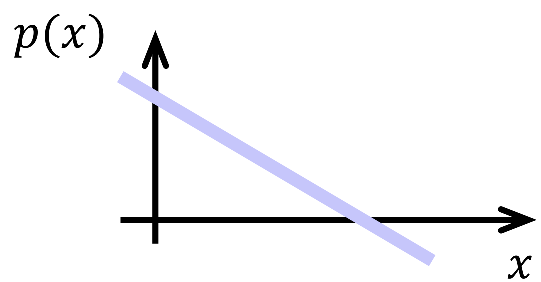
Roots of a Polynomial

A polynomial in variable x is a function of the form

degree

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \,.$$

- Number r is a **root** of $p(x)$ if $p(r) = 0$.
- A linear function has at most 1 root.



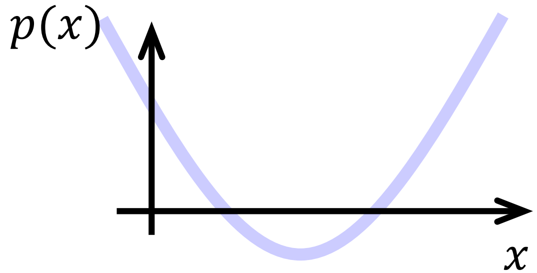
CS 237 Roots of a Polynomial

A polynomial in variable x is a function of the form

degree

$$p(x) = a_d x^{d} + a_{d-1}x^{d-1} + \cdots + a_0\,.$$

- Number r is a **root** of $p(x)$ if $p(r) = 0$.
- A quadratic function has at most 2 roots.



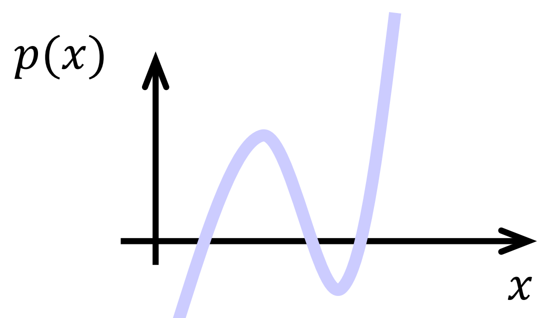
Roots of a Polynomial

A polynomial in variable x is a function of the form

degree

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \,.$$

- Number r is a **root** of $p(x)$ if $p(r) = 0$.
- A cubic function has at most 3 roots



Roots of a Polynomial

A polynomial in variable x is a function of the form

degree

$$p(x) = a_d x^{d} + a_{d-1} x^{d-1} + \cdots + a_0 \,.$$

- Number r is a **root** of $p(x)$ if $p(r) = 0$.

Fundamental Theorem of Algebra

A polynomial of degree d has at most d roots.

Verifying Polynomial Identities

- $(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \stackrel{?}{\equiv} x^6 - 7x + 37$

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

no use of randomness

Idea 1 (deterministic): Convert both polynomials to canonical form

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0.$$

- **It is slow:** If $f(x)$ is given as $(b_1 x - c_1) \cdot \ldots \cdot (b_d x - c_d)$ conversion by consecutively multiplying monomials requires about d^2 multiplications of coefficients.

(Faster with Fourier Transform)

End of lecture 1

- Next time, we will see a method for verifying polynomial identities using randomness.

Verifying Polynomial Identities

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Observation: Let $p(x) = f(x) - g(x)$.

Then we need to verify if $p(x) \equiv 0$.

Idea 2 (randomized): Evaluate $p(x)$ on random integers.

All numbers in this set are equally likely

Or, d could be any upper bound on the degree of $p(x)$, that is, $d \geq$ degree $p(x)$.

Let d = max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \ldots, 100d\}$.
2. Compute $p(r) = f(r) - g(r)$
3. **reject** if $p(r) = 0$; o. w. **accept**.

Only $d - 1$ multiplications needed to evaluate a product of d monomials

- Does this procedure accept or reject for our example when $r = 2$?

$$(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \stackrel{?}{\equiv} x^6 - 7x + 37$$

Analysis of Correctness

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Let d = max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \dots, 100d\}$.
2. Compute $p(r) = f(r) - g(r)$
3. **reject** if $p(r) = 0$; o. w. **accept**.

- If $f(x) \equiv g(x)$, we always accept.

No error in this case

- Otherwise, consider (non-zero) polynomial $p(x)$.
  - It has degree at most d.
  - By Fundamental Theorem of Algebra $p(x)$ has at most d roots.
  - We accept (incorrectly) only if we picked r to be a root of $p(x)$
  - This happens in at most d out of $100d$ cases, i.e., with probability at most

$$\frac{d}{100d} = \frac{1}{100}$$

.01 probability of error

How can we make it even smaller?