Probability in Computing

CS 237

LECTURE 2

Last time

- Course information
- Uses of probability in CS
- Verifying polynomial identities

Today

- Verifying polynomial identities
- Basic concepts in probability

Reminders

- Submit signed Collaboration and Honesty policy on Gradescope
- HW1 due Thursday

Last time: Verifying Polynomial Identities

- $(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \equiv^{?} x^6 - 7x + 37$

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

no use of randomness

- Simple deterministic methods for this task are slow.
- We learned a simple randomized method, based on:

Fundamental Theorem of Algebra

A polynomial of degree d has at most d roots.

Using Randomness for Verification

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Idea (randomized): Evaluate the polynomials on random integers.

All numbers in this set are equally likely

Or, d could be any upper bound on the degree of $p(x)$, that is, $d \geq$ degree $p(x)$.

Let $d =$ max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \ldots, 100d\}$.
2. Compute $f(r)$ and $g(r)$
3. **reject** if $f(r) \neq g(r)$; o. w. **accept**.

Only $d - 1$ multiplications needed to evaluate a product of d monomials

- Does this procedure accept or reject for our example when $r = 2$?

$$(x+1)(x-2)(x+3)(x-4)(x+5)(x-6) \equiv^{?} x^6 - 7x + 37$$

Top Hat question (Join Code: 033357)

What are the chances that our algorithm accepts on input

$$f(x) = 2x^2 + 2;$$
$$g(x) = \ x^2 + 3x \ ?$$

A. 100%

B. 50%

C. 2%

D. 1%

Let d = max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \ldots, 100d\}$.
2. Compute $f(r)$ and $g(r)$
3. **reject** if $f(r) \neq g(r)$; o. w. **accept**.

Analysis of Correctness

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Let d = max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \ldots, 100d\}$.
2. Compute $f(r)$ and $g(r)$
3. **reject** if $f(r) \neq g(r)$; o. w. **accept**.

- If $f(x) \equiv g(x)$, we always accept. *No error in this case*
- Otherwise, consider polynomial $p(x) = f(x) - g(x)$.
  - Checking if $f(x) \equiv g(x)$ is the same as checking if $p(x) \equiv 0$
  - Polynomial $p(x)$ has degree at most d; $p(x) \neq 0$ when $f(x) \neq g(x)$
  - By Fundamental Theorem of Algebra, $p(x)$ has at most d roots.
  - We accept (incorrectly) only if we picked r to be a root of $p(x)$
  - This happens in at most d out of $100d$ cases, i.e., with probability at most

$$\frac{d}{100d} = \frac{1}{100}$$

.01 probability of error

Reducing Probability of Error

Task: Given two polynomials $f(x)$ and $g(x)$, verify if $f(x) \equiv g(x)$.

Let $d =$ max degree of $f(x)$ and $g(x)$

1. Pick r uniformly from $\{1, \ldots, 100d\}$.
2. Compute $f(r)$ and $g(r)$
3. **reject** if $f(r) \neq g(r)$; o. w. **accept**.

- We accept (incorrectly) only if we picked r to be a root of $p(x)$
- This happens in at most d out of $100d$ cases, i.e., with probability at most

$$\frac{d}{100d} = \frac{1}{100}$$

.01 probability of error

How can we make it even smaller?

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Probability Theory

- Introduce three self-evident and indisputable properties of probability (the axioms)
- Develop the mathematical theory of probability from these axioms

Andrey Kolmogorov

[1903 - 1987]

"The theory of probability as a mathematical discipline can and should be developed from axioms in exactly the same way as geometry and algebra."

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Typical Statements about Probability

1. The probability that the randomized algorithm for checking polynomial identities accepts when the input is not an identity is at most 1/100.
2. The chance of getting a flush (that is, all cards of the same suit) in a 5-card poker hand is about 2 in 1000.
3. The chance of precipitation today in Boston is 20%.

Each such statement is implicitly talking about a random experiment

- either constructed by us, as in (1) and (2)
- or used by us to model the world, as in (3)

Probability: Random Experiment

- Random experiment: a repeatable procedure
  - Pick an integer from the set $\{1, \ldots, 100d\}$
  - Toss a coin
  - Toss a coin 3 times
  - Roll two dice
  - Pick a 5-card hand out of a deck of cards
  - Observe the number of goals in a soccer match between robots

Probability: Sample Space

- Outcome: result of the experiment
- Sample space Ω: set of all possible outcomes
  - Pick an integer from the set $\{1, \dots, 100d\}$

    $\Omega =$
  - Toss a coin

    $\Omega =$
  - Toss a coin 3 times

    $\Omega =$

Probability: Sample Space

- Outcome: result of the experiment
- Sample space Ω: set of all possible outcomes
  - Roll two dice

    Ω =

  - Pick a 5-card hand out of a deck of cards
    (the order of cards doesn't matter, so {**5♥, 5♦, 5♠, Q♥, Q♣**} and {**5♦, 5♥, Q♥, 5♠, Q♣**} are the same hand)

    Ω is the set of all subsets of five cards, |Ω| =
  - Observe the number of goals in a soccer match between robots

    Ω =

Probability: Event

- Event: a subset of the sample space

  (that is, a set of outcomes)

  – Experiment: toss a coin 3 times

    Event A : get at least 2 heads

    $A = \{$ $\}$

  – Experiment: roll two 6-sided dice

    Event B: the sum of the two numbers rolled is 11

    $B = \{$ $\}$

Top Hat question (Join Code: 033357)

- Experiment: toss a coin 3 times
- Which of the following is the event "exactly 2 heads"?

$$E_1 = \{HHT, HTH, THH, HHH\}$$

$$E_2 = \{HHT, HTH, THH\}$$

$$E_3 = \{HTH, THH\}$$

A. E_1

B. E_2

C. E_3

D. Both E_2 and E_3 are correct

Top Hat question (Join Code: 033357)

- Experiment: toss a coin 3 times
- Event $E = \{HTH, HHT, THH\}$

Which of the following describes the event E?

A. “exactly one head”
B. “exactly one tail”
C. “at most one tail”
D. None of the above

Events

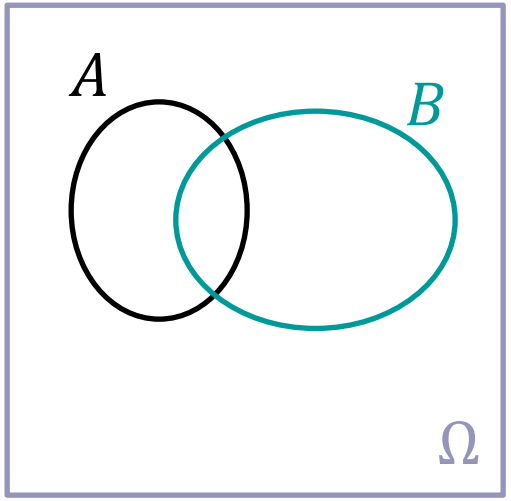
- Events are *sets* of outcomes
- We can combine events using set operations

$A \cap B$: the event that both A and B occurred

$A \cup B$: the event that A or B occurred

$A \setminus B$: the event that A occurred but B did not

$\overline{A}$: the event that A did not occur



Top Hat question (Join Code: 033357)

- Experiment: toss a coin 3 times

  Are the following events disjoint?

$A \cap B = \emptyset$

$A =$ "exactly 2 heads"

$B =$ "exactly 2 tails"

A. YES

B. NO

Top Hat question (Join Code: 033357)

- Experiment: toss a coin 3 times

$A \subseteq B$

Does the first event imply the second?

$A =$ "at least 2 heads"

$B =$ "exactly 2 heads"

A. YES

B. NO

Probability: Cast of Characters

- Random experiment: a repeatable procedure
- Outcome: result of the experiment
- Sample space Ω: set of all possible outcomes
- Event: a subset of the sample space
- Probability function Pr: assigns a probability Pr(E) to each event E

Probability Function

assigns a probability Pr(E) to each event E

- Experiment: toss a fair coin

E	∅	{H}	{T}	{H,T}
Pr(E)				

- Experiment: roll a die

Probability Function

assigns a probability Pr(E) to each event E

- Experiment: toss a fair coin 3 times

Probability Function

What principles did we use to come up with those probability functions?

- Symmetry: each outcome of the coin toss (or die roll) is equally likely
- The probability of each outcome is a number between 0 and 1
- Additivity: for events with more than one outcome, the probability of the event is the sum of the probabilities of its outcomes

Note: some experiments are not symmetric (toss a coin until H)