Probability in Computing

LECTURE 3

CS 237

Last time

- Verifying polynomial identities
- Basic concepts in probability

Today

- Axioms of probability
- How to compute probabilities
- Probability rules

Reminders

- Submit signed Collaboration and Honesty policy on Gradescope
- HW1 due tonight
- Prof. Sofya's OH: now in B21

Don't miss discussions tomorrow:
You will be making tricky dice.

Probability: Cast of Characters

- Random experiment: a repeatable procedure
- Outcome: result of the experiment
- Sample space Ω: set of all possible outcomes
- Event: a subset of the sample space
- Probability function Pr: assigns a probability Pr(E) to each event E

Probability Space
• **sample space Ω** together with • **probability function Pr**

Frequent example: uniform probability spaces,
where all outcomes have the same probability, $\frac{1}{|\Omega|}$.

Axioms of Probability

- Probability function assigns a probability to each event
- A probability function must satisfy the following properties, called axioms of probability
  - Non-negativity: $\Pr(E) \geq 0$ for all events $E \subseteq \Omega$
  - Additivity: if A and B are *disjoint* events then
    $$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

In particular, for each event E,
Pr(E) is the sum of probabilities of outcomes in E

if $A_1, A_2, \ldots$ are *disjoint* events then
$$\Pr(A_1 \cup A_2 \cup \cdots) = \Pr(A_1) + \Pr(A_2) + \cdots$$

  - Normalization: $\Pr(\Omega) = 1$

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Do we need to add
$\Pr(E) \leq 1$?

Top Hat question (Join Code: 413437)

- 50 students showed up to office hours today
- 20 are Red Sox fans (R), 25 are Patriots fans (P)
- Professor Sofya chooses a student uniformly at random.
- Let p be the probability that she chooses a Red Sox or a Patriot fan.

What is the range of possible values for p?

A. $p \geq 0.5$

B. $p \leq 0.4$

C. $0.4 \leq p \leq 0.9$

D. $0.5 \leq p \leq 0.9$

E. $0.4 \leq p \leq 0.5$

How Compute Probabilities

1. Find sample space
2. Define events of interest
3. Determine outcome probabilities
   - For uniform sample spaces: $\frac{1}{|\Omega|}$ for each outcome
4. Compute event probabilities
   - For uniform sample spaces: $\Pr(E) = \frac{|E|}{|\Omega|}$

Example: Dice

Roll two dice. What is the probability that

- the sum is at least 10?
- there is at least one 6?

$i \backslash j$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

1. $\Omega = \{(i,j): 1 \le i,j \le 6\}$
2. $A =$ event that sum is at least 10;

   $B =$ event that there is at least one 6
3. Uniform and $|\Omega| = 36$, so all outcomes have probability $\frac{1}{36}$
4. $\Pr(A) = \frac{|A|}{|\Omega|} =$

   $\Pr(B) = \frac{|B|}{|\Omega|} =$

Example: Card Shuffling

Shuffle a deck of 52 cards. What is the probability that all red cards come before all black cards?

1. Ω is the set of all permutations (orderings) of the deck

2. $E =$ event that all red cards come before all black cards;

3. Uniform and $|\Omega| = 52!$, so all outcomes have probability $\frac{1}{52!}$

4. $\Pr(E) = \frac{|E|}{|\Omega|} =$

Example: Poker Hands

Shuffle a deck of 52 cards and deal a 5-card hand (unordered). What is the probability that it is a flush (all cards have the same suit)?

1. Ω is the set of all poker hands; $|\Omega| = \binom{52}{5}$
2. E = event that it is a flush;

   $|E| = 4 \cdot \binom{13}{5}$
3. Uniform and $|\Omega| = \binom{52}{5}$, so all outcomes have probability $\frac{1}{\binom{52}{5}}$
4. $\Pr(E) = \frac{|E|}{|\Omega|} =$

Example: Fair Coins

Toss a fair coin n times. What is the probability that you get exactly $n/2$ heads?

1. Ω is the set of all sequences of H's and T's of length n
2. $E =$ set of all sequences with exactly $n/2$ heads.

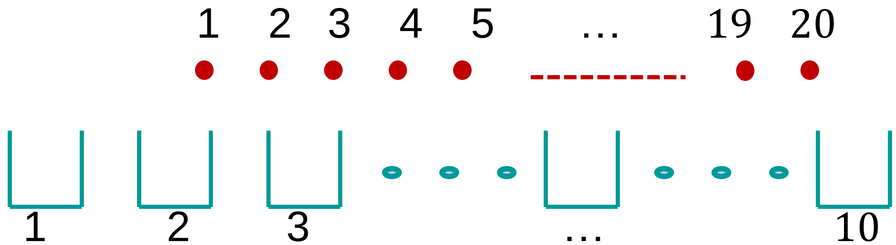
   $|E| = \binom{n}{n/2}$ if n is even
3. Uniform and $|\Omega| = 2^n$, so all outcomes have probability $\frac{1}{2^n}$
4. $\Pr(E) = \frac{|E|}{|\Omega|} =$

Stirling's approximation:
$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Example: Balls and Bins

Throw 20 labeled balls into 10 labeled bins. Each ball is equally likely to land in any bin, regardless of what happens to other balls.

What is the probability that bin 1 is empty?



1. $\Omega = \{(b_1, \ldots, b_{20}) : 1 \le b_i \le 10 \text{ for each } i = 1, \ldots, 20\}$

   $|\Omega| = 10^{20}$

2. $A =$ set of sequences as above, but $2 \le b_i \le 10$

3. Uniform and $|\Omega| = 10^{20}$, so all outcomes have probability $\frac{1}{10^{20}}$

4. $\Pr(E) = \frac{|A|}{|\Omega|} =$

Axioms of Probability

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  - Non-negativity: $\Pr(E) \geq 0$ for all events $E \subseteq \Omega$
  - Additivity: if A and B are *disjoint* events then
  $$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$
  - Normalization: $\Pr(\Omega) = 1$

Probability Rules

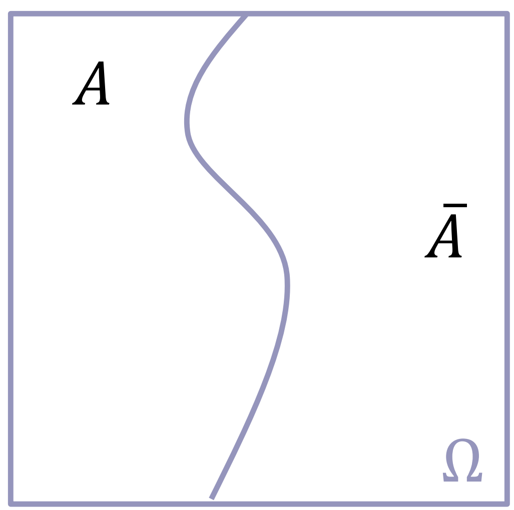
We can derive many useful properties from the axioms

- Complement rule: $\Pr(\bar{A}) = 1 - \Pr(A)$

  Proof: $\Pr(A) + \Pr(\bar{A}) = \Pr(A \cup \bar{A})$ *by additivity*

  $= \Pr(\Omega)$ *by definition of complement*

  $= 1$ *by normalization*



A and $\bar{A}$ are disjoint

Probability Rules

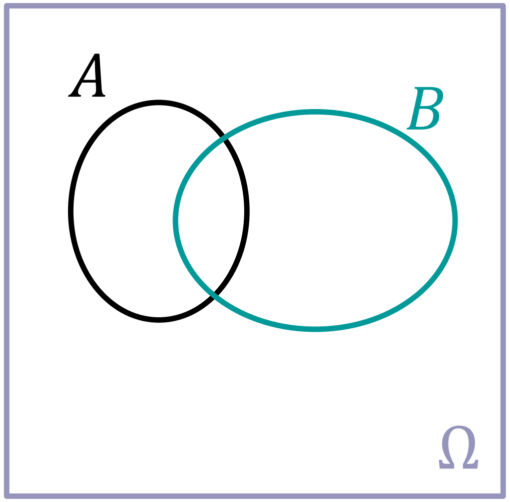
We can derive many useful properties from the axioms

- Difference rule: $\Pr(A/B) = \Pr(A) - \Pr(A \cap B)$

  Proof: $(A/B) \cup (A \cap B) = A$

  $\Pr(A/B) + \Pr(A \cap B) = \Pr(A)$

  by additivity



A/B and $A \cap B$ are disjoint

Probability Rules

We can derive many useful properties from the axioms

- Monotonicity rule: If $A \subseteq B$ then $\Pr(A) \leq \Pr(B)$

Proof: $A \cup (B/A) = B$

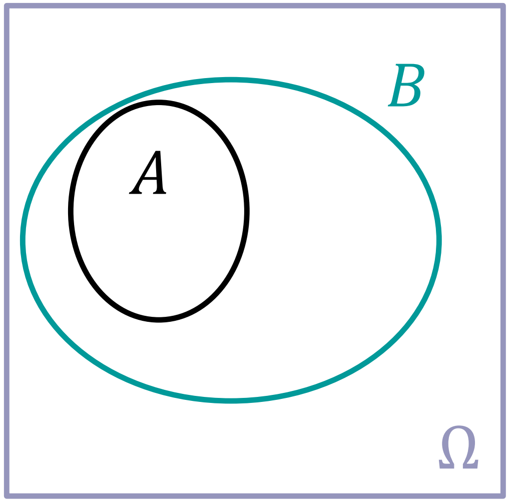
$$\Pr(A) + \Pr(B/A) = \Pr(B)$$

$$\Pr(A) = \Pr(B) - \Pr(B/A)$$

$$\leq \Pr(B)$$

by additivity

by non-negativity



A and $B \backslash A$ are disjoint

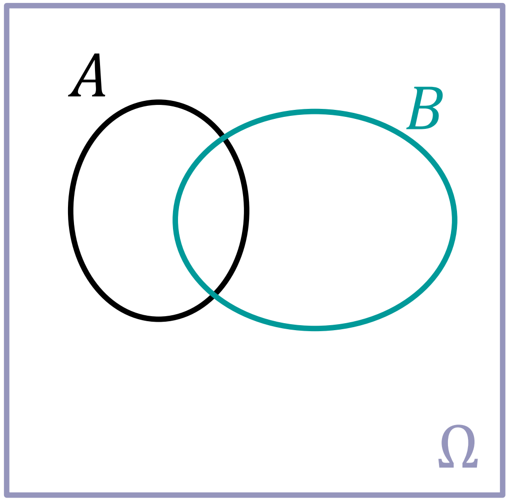
Probability Rules

We can derive many useful properties from the axioms

- Inclusion-Exclusion Principle:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

- Union Bound: $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$



Top Hat question (Join Code: 413437)

- For a bill to come before the US president, it must be passed by both the House and the Senate.
- Suppose that 40% of bills pass the House, 30% the Senate, and 50% pass at least one of the two.

What is the probability the next bill will come before the president?

A. 0.2

B. 0.4

C. 0.5

D. 0.7

E. None of the above

Picture source: ABC's Schoolhouse Rock