Probability in Computing

Lecture 5

Last time
• Common fallacies
• Tree diagrams
• Monty Hall problem

Today
• Spinners and Continuous Probability
• Geometric Probability
• Anomalies with Continuous Probability

Reminders
• HW2 due tonight
• Discussion 3 tomorrow
Suppose you have a bag containing Red balls and Green balls. You draw one ball from the bag, then a second ball. Let

- $R =$ a red ball was drawn
- $G =$ a green ball was drawn

The tree diagram shows the experiment. Consider the following three statements.

1. Drawing was without replacement
2. The bag originally contained 2 Red and 2 Green balls
3. $\Pr(\text{the 2 balls were of different colors}) = \frac{2}{3}$

A. Only statement (1) is true.  
B. Only statement (2) is true.  
C. Only (2) and (3) are true.  
D. All three statements are true.
The Spinner Experiment

Assumptions:
- The circumference of the circle is 1.0
- When you spin the pointer, it is equally likely to end up anywhere around the circle
Continuous Probability

The Spinner Experiment

- **Experiment**: spin the pointer, see where it lands!
- **Outcome**: real number $x \in [0,1]$.
- **Sample space**: $\Omega = [0,1]$. 

Sofya Raskhodnikova, Wayne Snyder; Probability in Computing
The Spinner Experiment

This is implemented by the numpy function `random()`!

```python
from numpy.random import random
for k in range(5):
    print(random())
```

```
0.288092875517823
0.7162125723051093
0.6655015245054124
0.554162601188949
0.28532267939275413
```

\[ 0.359507900573786, 0.43703195379934145, 0.6976311959272649, 0.06022547162926983, 0.6667667154456677 \]
What is the probability that the dial lands in [0, 1/8]?

(A) 1/8  
(B) 0  
(C) 0.01  
(D) none of the above
Continuous Probability

- **Experiment:** spin the pointer, see where it lands!
- **Outcome:** real number \( x \in [0,1) \)
- **Sample space:** \( \Omega = [0,1) \)
- **Probability Function:**

\[
Pr(x \in [a,b]) = \frac{\text{length of } [a,b]}{\text{length of } [0,1)} = b - a
\]

\([a, b] \subseteq [0,1)\)
Uniform Sampling from an Interval \([a..b]\)

- **Experiment**: Sample a point uniformly from \([a..b]\)
- **Outcome**: real number \(x \in [a, b]\)
- **Sample space**: \(\Omega = [a, b]\)
- **Probability Function**:

\[
Pr(x \in [c, d]) = \frac{\text{length of } [c, d]}{\text{length of } [a, b]} = \frac{d - c}{b - a}
\]
Experiment: choose uniformly $x \in [0, 10]$

$$P_{r}(x \in [0,6]) =$$

$$P_{r}(x \in [1,6] \cup [8,10]) =$$

$$P_{r}(x \in [0,6] \cup [5,7]) =$$

```python
from numpy.random import uniform

for k in range(5):
    print(uniform(0,10))
```

9.458670834431103
7.821264914591
4.367437002029626
1.1280161736707706
0.4086456784560122
Continuous Probability

Let’s kick it up a notch! What about 2d?

- **Experiment:** Choose a random point in unit square, all points equally likely
- **Event:** hitting point inside rectangle \( P \)
- **Sample space:** the square
- **Probability function:**

\[
\Pr[p \in P] = \frac{\text{area of } P}{\text{area of square}}
\]
def random_plane_plot(num_trials):
    x_vals = [random() for k in range(num_trials)]
    y_vals = [random() for k in range(num_trials)]
    plt.figure(num=None, figsize=(10, 10))
    plt.title('Graph of Uniform Distribution of '+str(num_trials)+' points in 2D.', fontsize=12)
    plt.grid(color='0.95')
    plt.ylim(0, 1)
    plt.xlim(0, 1)
    plt.scatter(x_vals, y_vals, marker="o", color="r")
    plt.show()

random_plane_plot(30)
Suppose you throw a dart at a square target 1 meter on a side. What is the probability that it lands within 0.1 m of an edge?

A. $0.8^2 = 0.64$
B. $0.9^2 = 0.81$
C. $1 - 0.8^2 = 0.36$
D. $1 - 0.9^2 = 0.19$
Why stop at 2d?

- **Experiment**: Choose a random point in 2x2x2 cube
- **Event**: hitting point inside unit sphere in cube
- **Sample space**: the cube
- **Probability function**:

\[
\Pr[p \in ] = \frac{\text{volume of sphere}}{\text{volume of cube}}
\]
Continuous Probability
Here’s a (somewhat) realistic problem:

- Wayne and Sofya decide to meet at Starbucks between noon and 1pm to discuss HW 03.
- Each of them arrives at a time chosen uniformly at random between noon and 1pm.
- Once one of them arrives, they will wait for the other for 15 minutes and leave if they do not show up.
- What is the probability that Wayne and Sofya meet?
Continuous Probability
Anomalies with Continuous Probability

Let’s go back to the spinner experiment:

What is the probability that the pointer lands exactly on 0.25?

(A) $p = 0$  (B) $p > 0$
Anomalies with Continuous Probability

Let’s go back to the spinner experiment:

What is the probability that the pointer lands exactly on 0.25?

(A) \( p = 0 \)  \hspace{1cm}  (B) \( p > 0 \)
Continuous Probability

- What is the probability that the dial lands on 1/4?
- Suppose we suspect that it is, say, 0.1
- Pick 11 distinct points on the circle

\[
Pr(\text{lands on } x_i) = 0.1 \text{ for all } i
\]

by symmetry:

\[
Pr(\text{lands on one of } x_1, ..., x_{11}) = 11 \cdot 0.1 = 1.1
\]

contradiction!
Consider the spinner one more time. We decided that

$$\Pr( x \in [0, 1/4] ) = 1/4$$

But what is

$$\Pr( x \in [0, 1/4) ) = ??$$

$$[a, b] = \{ x \mid a \leq x \leq b \}$$

$$[a, b) = \{ x \mid a \leq x < b \}$$

A. $1/4 - \epsilon$
B. $1/4$
C. The probability does not exist
• **Discrete** sample spaces: outcomes are countable
  – Toss a coin, roll two dice, toss a coin until we get heads, …

• **Continuous** sample spaces: outcomes are not countable
  – Spinner, darts, time, space, ….
Continuous Probability

- **Discrete** sample spaces: outcomes are countable
  - We assign a probability to each outcome
  - The probability of an event is the sum of the probabilities of the outcomes comprising the event

- **Continuous** sample spaces: outcomes are not countable
  - Individual outcomes have probability zero
  - We assign probabilities to the extent (length, area, volume, etc.) taken up by the event of interest