Probability in Computing

LECTURE 5

CS 237

Last time

- Common fallacies
- Tree diagrams
- Monty Hall problem

Today

- Continuous probability spaces
- Geometric probability method
- Anomalies with Continuous Probability

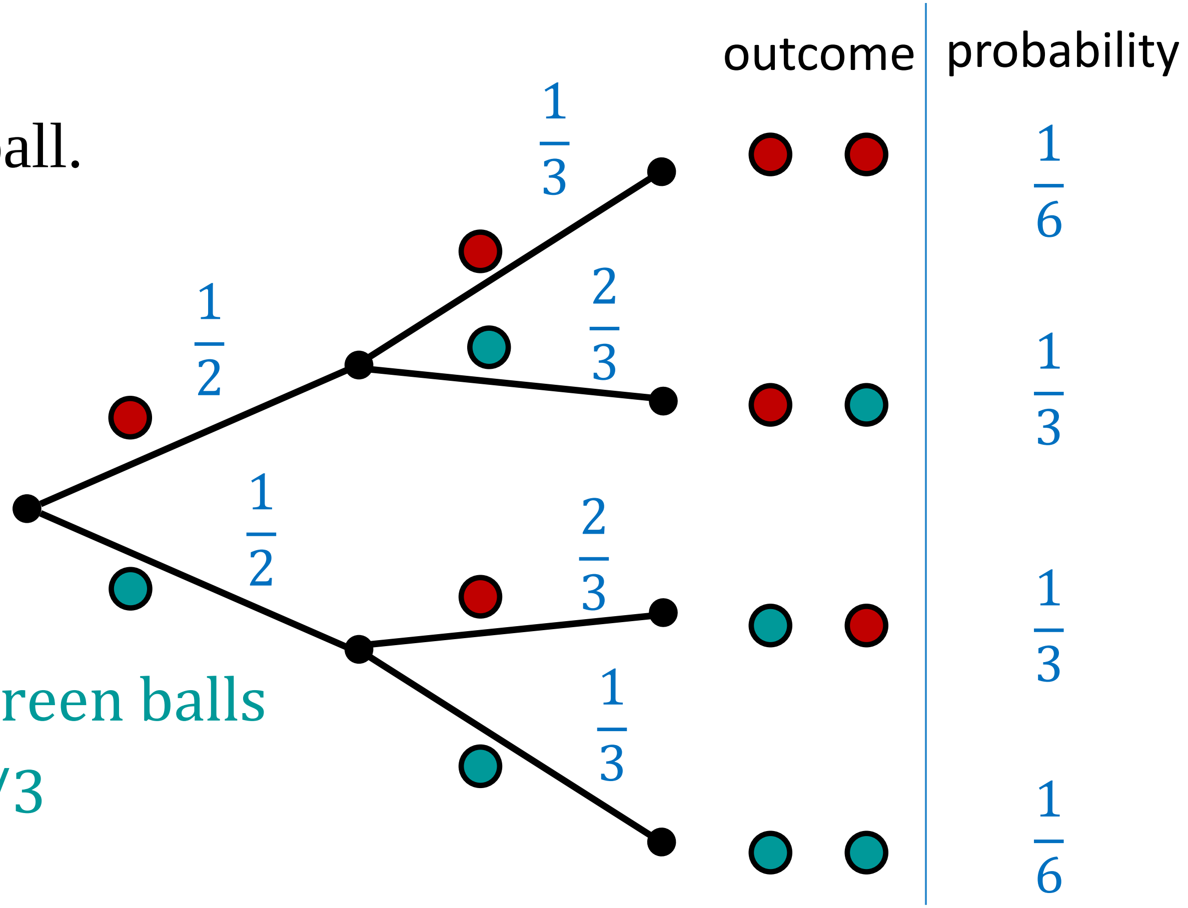
Reminders

- HW2 due tonight
- Discussion 3 tomorrow

Top Hat question (Join Code: 033357)

You have a bag containing red and green balls.
You draw one ball from the bag, then a second ball.

The tree diagram shows the experiment.



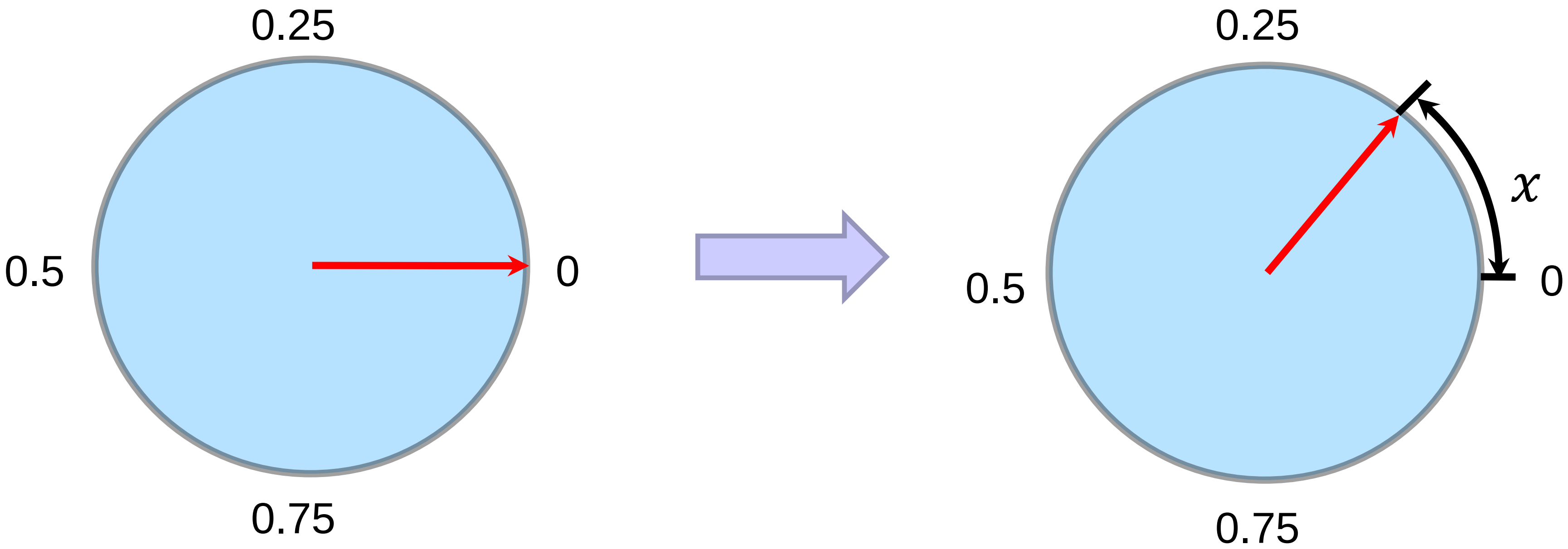
Consider the following three statements.

1. Drawing was without replacement
2. The bag originally contained 2 red and 2 green balls
3. Pr(the 2 balls are of different colors) = 2/3

A. Only statement (1) is true.
B. Only statement (2) is true.
C. Only (2) and (3) are true.
D. All three statements are true.

Continuous Probability Spaces

The Spinner Experiment



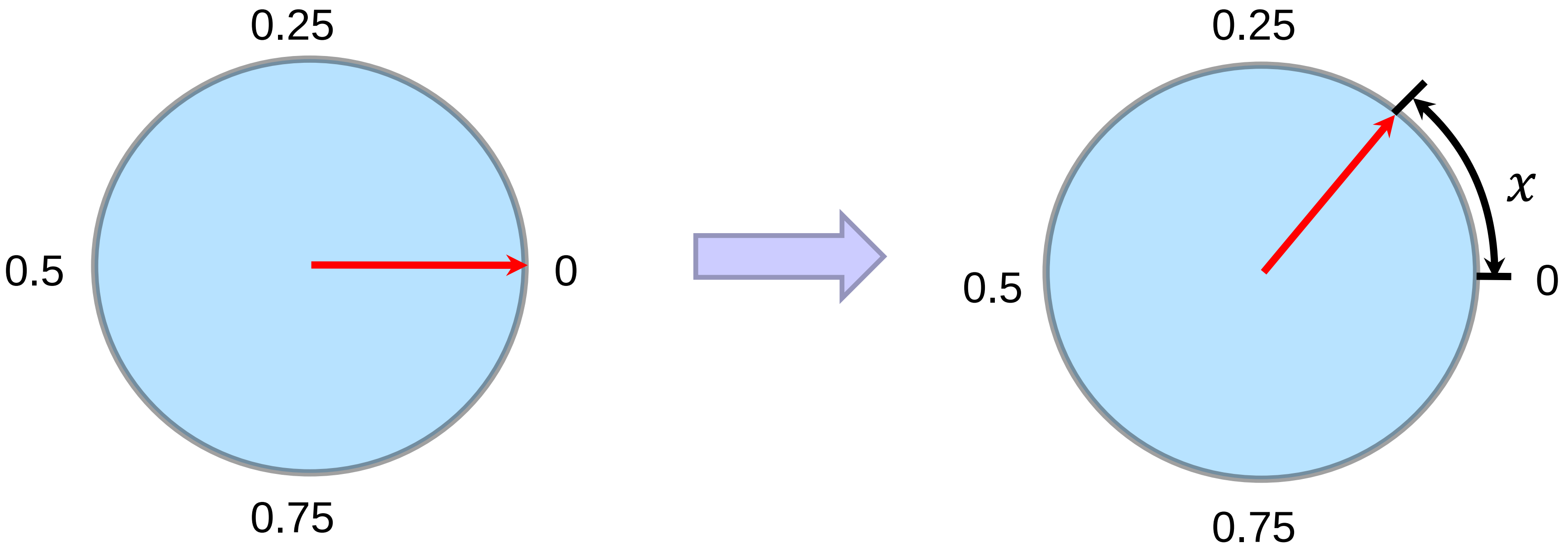
Assumptions:

– The circumference of the circle is 1

– When you spin the pointer, it is equally likely to end up anywhere around the circle

Continuous Probability Spaces

The Spinner Experiment



- Experiment: spin the pointer, see where it lands!
- Outcome: real number $x \in [0,1]$
- Sample space: $\Omega = [0,1]$

Continuous Probability Spaces

The Spinner Experiment

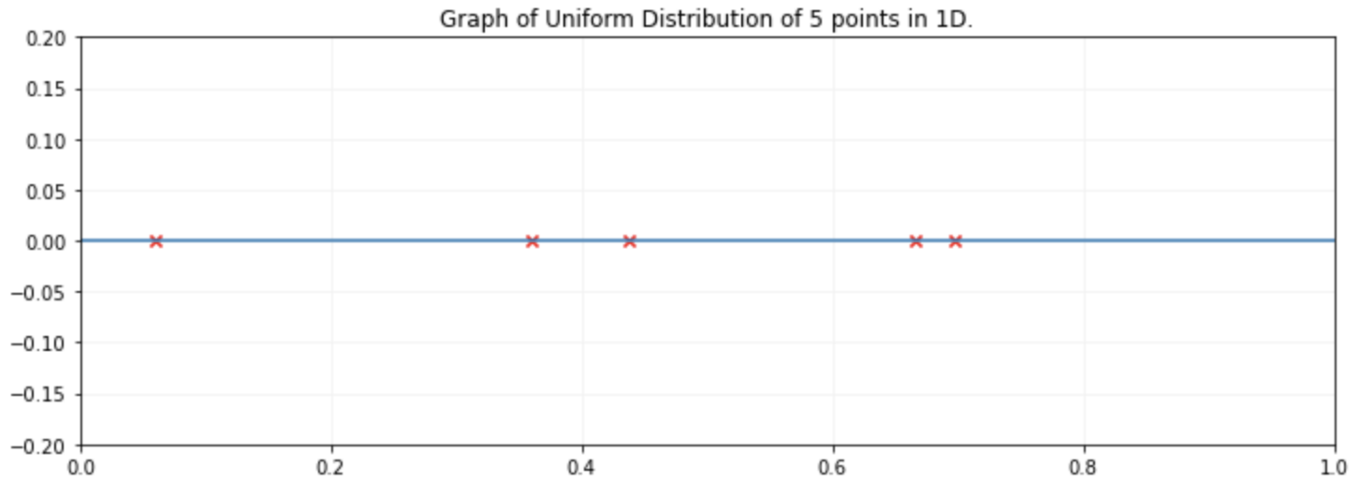
This is implemented by
the numpy function random()

```
from numpy.random import random

for k in range(5):
    print(random())
```

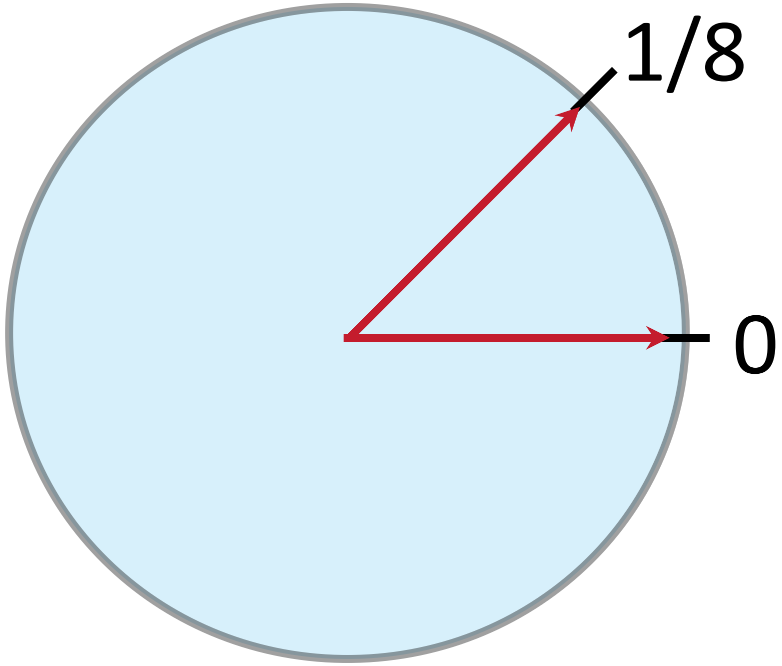
```
0.2880928755517823
0.7162125723051093
0.6655015245054124
0.554162601188949
0.2853226793927541 3
```

```
[0.359507900573786, 0.43703195379934145, 0.6976311959272649, 0.06022547162926983, 0.6667667154456677]
```



Question

What is the probability that the dial lands in [0, 1/8]?

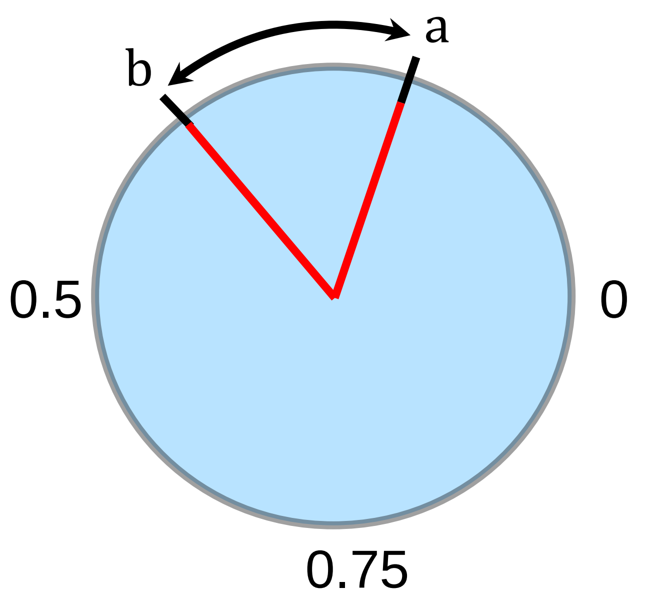


A. 1/8
B. 0
C. 0.01
D. None of the above

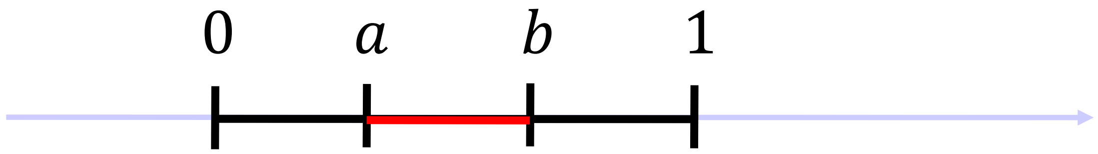
Uniform Sampling from [0, 1]

- Experiment: spin the pointer, see where it lands!
- Outcome: real number $x \in [0,1]$
- Sample space: $\Omega = [0,1]$
- Probability function:

$$\Pr(x \in [a,b]) = \frac{\text{length of } [a,b]}{\text{length of } [0,1]} = b - a$$



$$[a,b] \subseteq [0,1]$$

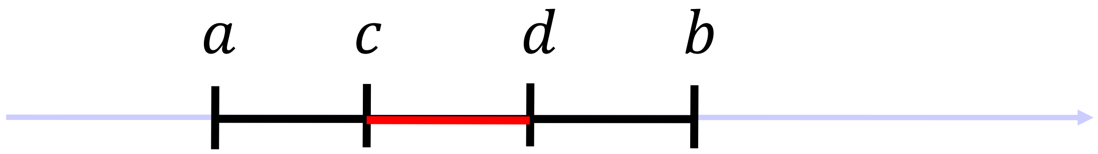


Uniform Sampling from an Interval $[a, b]$

- Experiment: sample a point uniformly from $[a, b]$
- Outcome: real number $x \in [a, b]$
- Sample space: $\Omega = [a, b]$
- Probability function:

$$\Pr(x \in [c, d]) = \frac{\text{length of } [c, d]}{\text{length of } [a, b]} = \frac{d - c}{b - a}$$

$$[c, d] \subseteq [a, b]$$



CS 237

Continuous Probability

Experiment: choose uniform $x \in [0,1]$


```python
from numpy.random import uniform

for k in range(5):
    print(uniform(0,10))
```

```
9.458670834431103
7.821264914591
4.367437002029626
1.1280161736707706
0.4086456784560122
```

$\Pr(x \in [0,6]) =$

$\Pr(x \in [1,6] \cup [8,10]) =$

$\Pr(x \in [0,6] \cup [5,7]) =$

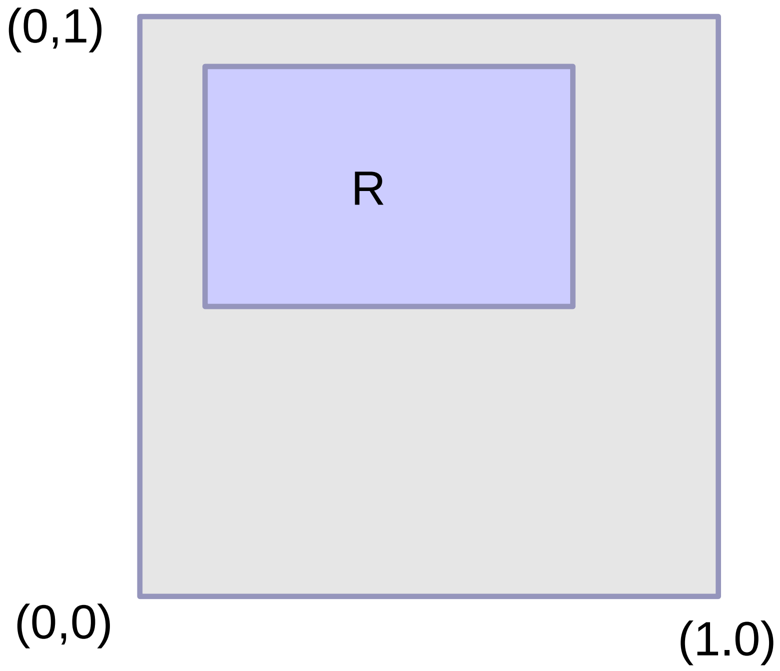
$\Pr(x \in [-10,5]) =$

Continuous Probability Spaces

Let's kick it up a notch! What about 2 dimensions?

- Experiment: Choose a uniformly random point x in unit square
- Event: x lands inside region R
- Sample space: the square
- Probability:

$$\Pr[p \in R] = \frac{area\ of\ R}{area\ of\ square}$$



Uniform Sampling from Rectangles

```python
# uniform sampling from unit square [0, 1]x[0, 1]
def uniform_unit_square():
    x = uniform(0, 1)
    y = uniform(0, 1)
    return (x, y)

# uniform sampling from the rectangle [a, b]x[c, d]
def uniform_rectangle(a, b, c, d):
    x = uniform(a, b)
    y = uniform(c, d)
    return (x, y)
```

Top Hat question (Join Code: 033357)

You throw a dart at a square target 1 meter on a side.

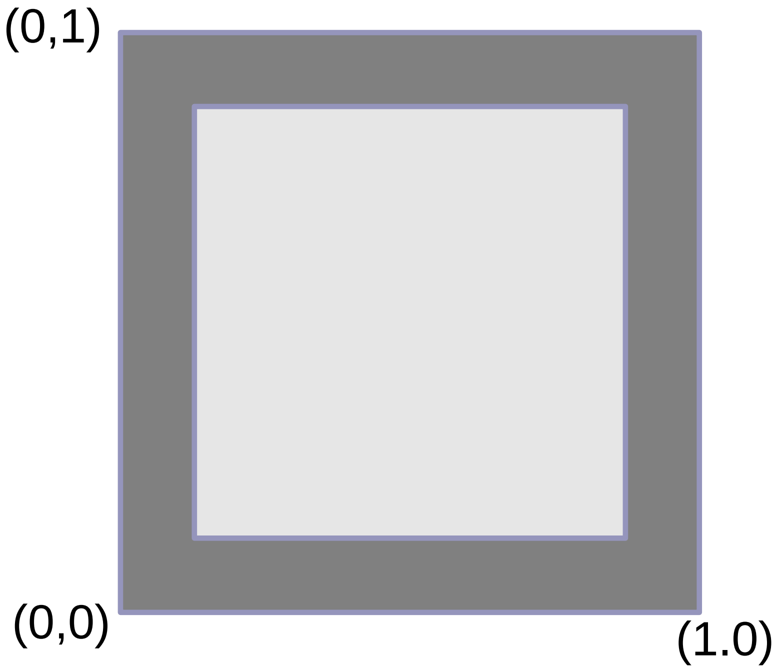
What is the probability that it lands within 0.1 m of an edge?

A. $0.8^2 = 0.64$

B. $0.9^2 = 0.81$

C. $1 - 0.8^2 = 0.36$

D. $1 - 0.9^2 = 0.19$

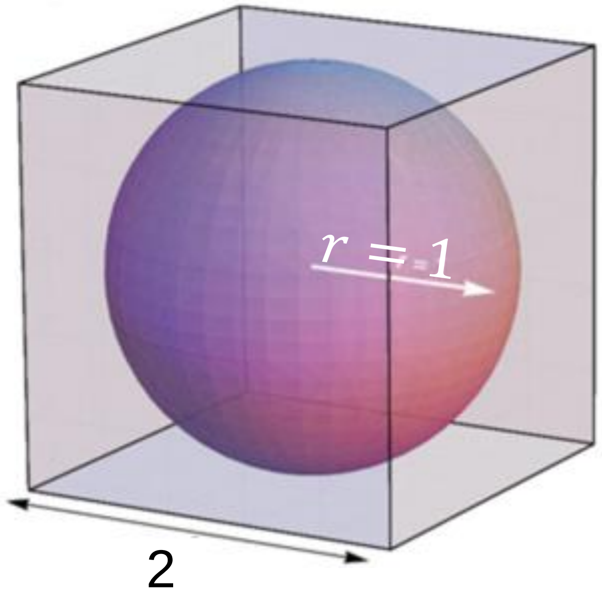


Continuous Probability

Why stop at 2 dimensions?

- Experiment: Choose a uniformly random point x in 2x2x2 cube
- Event: x lands inside the unit sphere
- Sample space: the cube
- Probability:

$$\Pr[x \in sphere] = \frac{volume\ of\ sphere}{volume\ of\ cube}$$



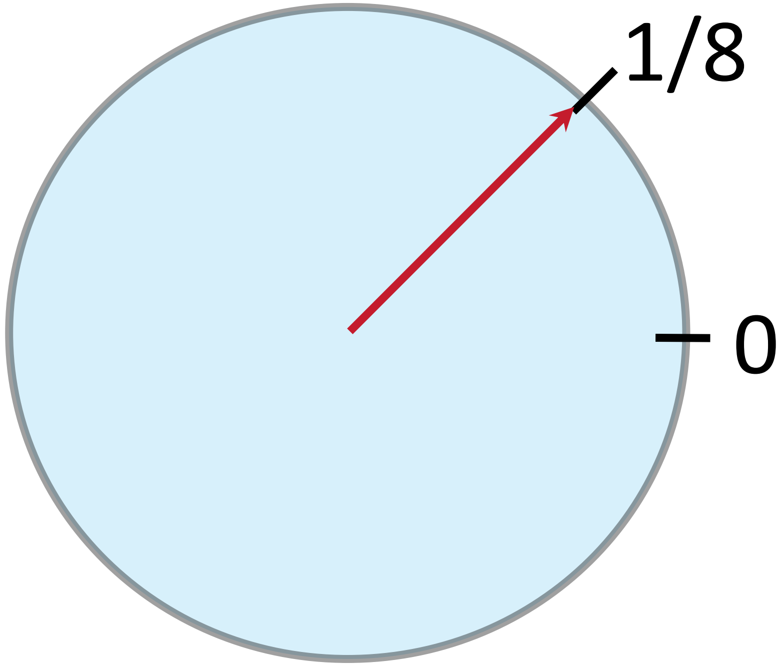
Randomly Tardy Problem

- Tiago and Sofya decide to meet at Starbucks between noon and 1pm to discuss HW 03.
- Each of them arrives at a time chosen uniformly at random between noon and 1pm.
- Once one of them arrives, they will wait for the other for 15 minutes and leave if they do not show up.
- What is the probability that Tiago and Sofya meet?

Randomly Tardy Problem

Question

What is the probability that the dial lands on 1/8?



A. 1/8

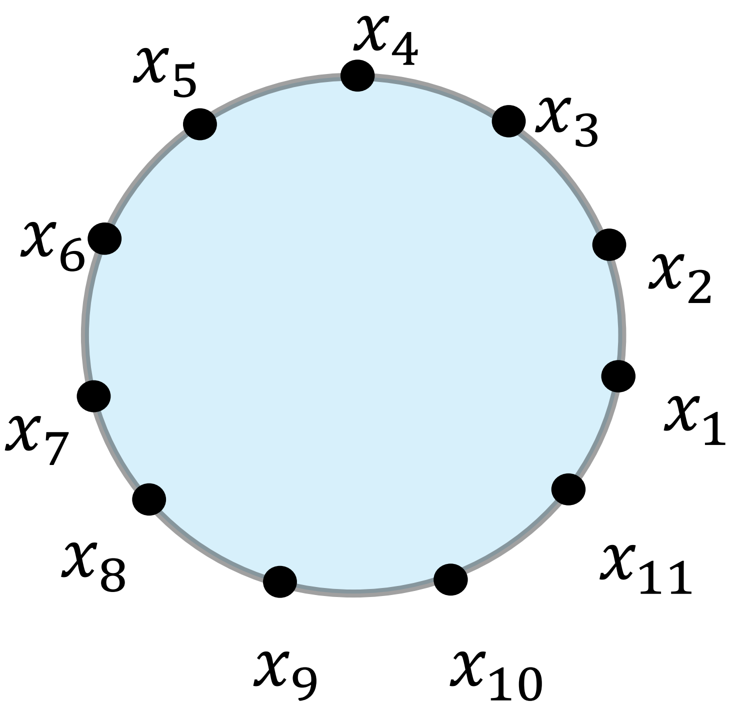
B. 0

C. 0.01

D. None of the above

Anomalies with Continuous Probability Spaces

- What is the probability that the dial lands on 1/8?
- Suppose we suspect that it is, say, 0.1
- Pick 11 distinct points on the circle



by symmetry: $\Pr(lands\ on\ x_i) = 0.1\ for\ all\ i$

by additivity: $\Pr(lands\ on\ one\ of\ x_1, \ldots, x_{11}) = 11 \cdot 0.1 = 1.1$

Contradiction!

Anomalies with Continuous Probability Spaces

Let p be the probability of an outcome, e.g., 1/8.

We can show that $p = 0$ by contradiction:

- Suppose $p \neq 0$ for the sake of contradiction.
- Pick distinct points $x_1, \ldots, x_k$, where $k > 1/p$

by symmetry: $\Pr(\text{dial lands on } x_i) = p$ for all $i \in \{1,2,\ldots,k\}$

by additivity: $\Pr(\text{dial lands on one of } x_1, \ldots, x_k) = k \cdot p > 1$

Contradiction!

CS 237 Top Hat question (Join Code: 033357)

Consider the spinner one more time.... We decided that

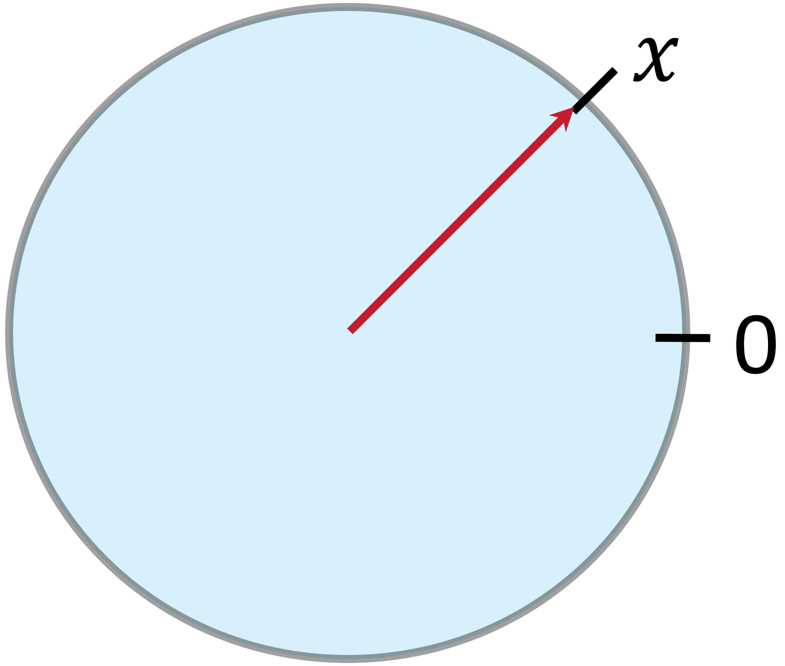
$$\Pr(x \in [0,1/4]) = 1/4$$

But what is

$$\Pr(\, x \in [0, 1/4) \,) = ??$$

$$[a, b] = \{\, x \mid a \le x \le b \,\}$$

$$[a, b) = \{\, x \mid a \le x < b \,\}$$



A. $1/4 - \epsilon$

B. $1/4$

C. The probability does not exist

Probability Spaces

- **Discrete sample spaces:** outcomes are countable
  - Examples of random experiments modeled by them:
    Toss a coin, roll two dice, toss a coin until we get heads, …
  - We assign a probability to each outcome
  - **The probability of an event** is the sum of the probabilities of the outcomes comprising the event
- **Continuous sample spaces:** outcomes are not countable
  - Examples of random experiments modeled by them:
    Spinner, darts, meeting at a random time or in a random location, ...
  - Individual outcomes have probability zero
  - **The probability of an event** is the proportion of the extent (length, area, volume, etc.) taken up by it