Probability in Computing

LECTURE 6

Last time
- Spinners and Continuous Probability
- Geometric Probability
- Anomalies with Continuous Probability

Today
- Five kinds of sample spaces
- Random Variables

Reminders
- HW 3 due on Thursday
- A PDF of this lecture is in Piazza Resources
Discrete Sample Spaces are finite or countably infinite
- We assign a probability to each outcome
- The probability of an event is the sum of the probabilities of its members: \( A = \{a_1, \ldots, a_n\} \)
  \[ \Pr(A) = \Pr(a_1) + \cdots + \Pr(a_n) \]
- Can be uniform or non-uniform

1. Finite Uniform Sample Spaces: Example

- Experiment: Roll a die, count the number of dots showing
- Sample space: \( \Omega = \{1, 2, 3, 4, 5, 6\} \)
- Probability Function:

\[
\Pr(A) = \frac{|A|}{|\Omega|}
\]
Summary: 5 Types of Sample Spaces

2. Finite Non-Uniform Sample Spaces: Example

- **Experiment:** Throw two dice, count the number of dots showing
- **Sample space:** \( \Omega = \{ 2, 3, 4, \ldots, 12 \} \)
- **Probability Function:**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Pr}(\omega) )</td>
<td>( \frac{1}{36} )</td>
<td>( \frac{2}{36} )</td>
<td>( \frac{3}{36} )</td>
<td>( \frac{4}{36} )</td>
<td>( \frac{5}{36} )</td>
<td>( \frac{6}{36} )</td>
<td>( \frac{5}{36} )</td>
<td>( \frac{4}{36} )</td>
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<td>( \frac{2}{36} )</td>
<td>( \frac{1}{36} )</td>
</tr>
</tbody>
</table>

![Probability distribution for dots showing on two dice](image)
3. Countably Infinite Non-Uniform Sample Spaces: Example 1

- **Experiment:** Flip a coin until a head, count how many flips.
- **Sample space:** \( \Omega = \{ 1, 2, 3, 4, 5, \ldots \} \)
- **Probability Function:**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( k )</th>
<th>( 1/2^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pr(\omega) )</td>
<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>1/16</td>
<td>( \ldots )</td>
<td>( 1/2^k )</td>
</tr>
</tbody>
</table>

![Probability Distribution for Flipping until a Head](image)
3. Countably Infinite Non-Uniform Sample Spaces: Example 2

- **Experiment:** Roll a die until a 6 appears, count how many rolls.
- **Sample space:** \( \Omega = \{ 1, 2, 3, 4, \ldots \} \)
- **Probability Function:**

\[
\begin{array}{cccccc}
\omega & 1 & 2 & 3 & \ldots & k & \ldots \\
Pr(\omega) & 1/6 & 5/36 & 25/216 & \ldots & 5^{k-1}/6^k & \ldots
\end{array}
\]
Continuous sample spaces are not countable
- Individual outcomes have probability zero
- We assign probabilities to the extent (length, area, volume, etc.) taken up by the event of interest

4. Uncountably Infinite Uniform Sample Spaces: Example

- **Experiment:** Uniformly pick a real number from 0 to 1.
- **Sample space:** $\Omega = [0..1]$
- **Event:** $\omega \in [a..b]$
- **Probability Function:** $Pr( [a..b] ) = b - a$
5. Uncountably Infinite Non-Uniform Sample Spaces: Example

- **Experiment**: Throw a dart at a circular target of radius 1.0, what is the distance from the center?
- **Sample space**: \( \Omega = [0..1] \)
- **Event**: \( \omega \in [a..b] \)
- **Probability Function**:

\[
\Pr([a..b]) = \frac{\pi b^2 - \pi a^2}{\pi 1^2} = b^2 - a^2
\]
Experiment: Pick a BU student uniformly at random and measure their height in meters, rounded to 2 decimal places.

The Sample Space is:

A. Finite and uniform
B. Finite and not uniform
C. Countably infinite
D. Uncountably infinite and uniform
E. Uncountably infinite and not uniform
A **Random Experiment** is a repeatable procedure that produces uncertain outcomes from a well-defined sample space.

Example: Flip a coin!
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**Example:** Flip a coin!
Random Variables

**Definition.** A Random Variable $X$ is a function from a sample space $\Omega$ into the reals:

$$X : \Omega \rightarrow \mathbb{R}$$

It *interprets* the random experiment as a real number. Now when an outcome is requested, the sample point is translated into a real number:

$$X = \text{“Flip a coin and count the number of heads showing”}$$
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$$X = \text{“Flip a coin and count the number of heads showing”}$$
A random variable will not in general simply rename outcomes with real numbers, it will compute some useful information about the outcome.

\[ V = \text{"Flip 3 coins: return 1 if there are more heads showing than tails, 0 if more tails than heads"} \]

\[ \Omega = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\} \]

\[ V \simulates a vote among three coins for H or T! \]
A random variable will not in general simply rename outcomes with real numbers, it will compute some useful information about the outcome.

\[ V = \text{“Flip 3 coins: return 1 if there are more heads showing than tails, 0 if more tails than heads”} \]
Joint Random Variables are combinations of 2 or more RVs, giving *multiple interpretations* of the same experiment.

Example (from LLM):
“Flip 3 coins and
(C) return the number of heads showing;
(M) return 1 if all heads or all tails, 0 otherwise”
Joint Random Variables are combinations of 2 or more RVs, giving *multiple interpretations* of the same experiment.

**Example (from LLM):**

“Flip 3 coins and
(C) return the number of heads showing;
(M) return 1 if all heads or all tails, 0 otherwise”

![Diagram of Joint Random Variables]
Consider the following experiment:
“Flip 3 coins and

(T) return the number of tails showing;
(V) return 1 if there are more heads showing than tails, 0 if more tails than heads

Suppose the random experiment chooses HHT from Ω; then the result is:

A. T = 1  V = 0
B. T = 2  V = 0
C. T = 2  V = 1
D. None of the above
The term “Random Variable” is a misnomer: it’s really a function!

Two things to remember about RVs as functions:

1. A random variable $X$ is characterized by its range:

   \[ \text{Range}(X) = \{ r \mid \exists \omega \in \Omega. X(\omega) = r \} \]

$X$ is finite/countably infinite/continuous/uniform/non-uniform if it behaves that way on its range.

Example: Choose a student from this class:

- $X =$ “Their height rounded to the nearest mm;” (Finite/non-uniform)
- $Y =$ “Their age in seconds;” (Continuous/non-uniform)
- $Z =$ “How many tattoos they have” (Finite/non-uniform)
Second thing to remember about random variables as functions:

2. Random variables behave just like Python functions which return random values.

X = “Flip a coin and count the number of heads showing”

```
In [1]: 1 from numpy.random import randint
2
3 def X():
4    return randint(2)
5
6 X()
```

Out[1]: 0

```
In [2]: 1 X()
```

Out[2]: 1
“Flip 3 coins and
(C) return the number of heads showing;
(M) return 1 if all heads or all tails, 0 otherwise”

```
In [3]:
    def CM():
        omega = (X(), X(), X())
        print("outcome = ", omega)
        C = sum(omega)
        M = (1 if C == 3 or C == 0 else 0)
        return (C,M)

    CM()

outcome =  (1, 0, 0)

Out[3]:  (1, 0)

In [22]:
    CM()

outcome =  (1, 1, 1)

Out[22]:  (3, 1)
```
A very typical notion for probabilities involving a random variable \( X \) is:

\[
\Pr( < \text{some Boolean expression involving } X > )
\]

Examples for discrete RVs:

\[
\begin{align*}
\Pr(X = k) &= P_X(k) \\
\Pr(X \neq k) &= 1.0 - P_X(k) \\
\Pr(X \leq k) &= \sum_{a \leq k} P_X(a) \\
\Pr(j < X \leq k) &= \sum_{j < a \leq k} P_X(a)
\end{align*}
\]

A useful notation in the discrete case from the alternate textbook:

\( P_X(k) = \) the probability that \( X \) returns \( k \)
Example:

\[ X = \text{“Count the number of flips until the coin shows heads”} \]

\[ \text{Range}(X) = \{ 1, 2, 3, 4, \ldots \} \]

\[ Pr(X = k) = \begin{cases} 2^{-k} & k > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ Pr(X = 2) = \frac{1}{4} \]

\[ Pr(X \leq 4) = \frac{15}{16} \]

\[ Pr(2 \leq X < 4) = \frac{3}{8} \]

\[ Pr(X \text{ is odd}) = \frac{2}{3} \]
For the random variable

\[ X = \text{“Count the number of flips until the coin shows heads”} \]

What is \( \Pr(X > k) \) for \( k > 0 \)?

A. \( \frac{1}{k} \)

B. \( \frac{1}{2^k} \)

C. \( \frac{1}{2^{k+1}} \)

D. \( \frac{1}{2^{k-1}} \)
Functions of Random Variables

New random variables can be created by functions or expressions involving old random variables. But you have to be careful!

Example: \( X = \) “the number of dots on a thrown die”

\[
\begin{align*}
\text{Range}(X) &= \{ 1, 2, 3, 4, 5, 6 \} \\
\Pr(X = k) &= \frac{1}{6} \text{ for } k \in \text{Range}(X) \\
&= 0 \text{ otherwise}
\end{align*}
\]

Let \( Y = X + 10 \)

\[
\begin{align*}
\text{Range}(Y) &= \{ 11, 12, 13, 14, 15, 16 \} \\
\Pr(Y = k) &= \frac{1}{6} \text{ for } k \in \text{Range}(Y) \\
&= 0 \text{ otherwise}
\end{align*}
\]
Functions of Discrete Random Variables

Let $Z = 2 \times X$

Range($Z$) = \{ 2, 4, 6, 8, 10, 12 \}

$Pr(Z = k) = 1/6 \text{ for } k \in \text{Range}(Z)$

0 \text{ otherwise}

Let $Q = 2^X$

Range($Q$) = \{ 2, 4, 8, 16, 32, 64 \}

$Pr(Q = k) = 1/6 \text{ for } k \in \text{Range}(Q)$

0 \text{ otherwise}

Note: The probabilities have not changed (so far...).
Why did I say you have to be careful? Two main reasons...

One, the function of a random variable may combine outcomes...

Example:

Let $S = X - 3$ and let $T = |X - 3|$

Range(S) = \{-2, -1, 0, 1, 2, 3\}

Pr(S = k) = \frac{1}{6}$ for k $\in$ Range(S)

0 otherwise

Range(T) = \{0, 1, 2, 3\}

Pr(T = k) = \frac{1}{6}$ for k $\in$ \{0, 3\}

\frac{1}{3}$ for k $\in$ \{1, 2\}

0 otherwise
Two, you have to be careful when a random variable is used more than once, since each occurrence refers to a potentially different random outcome! Compare:

Let \( Z = 2 \times X \) (twice the dots showing on a single die)

\[
\text{Range}(Z) = \{ 2, 4, 6, 8, 10, 12 \}
\]

\[
\Pr(Z = k) = \frac{1}{6} \quad \text{for } k \in \text{Range}(Z)
\]

\[
0 \quad \text{otherwise}
\]

Let \( D = X + X \) (sum of the dots showing on two thrown dice)

<table>
<thead>
<tr>
<th>Range(D)</th>
<th>2</th>
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<th>5</th>
<th>6</th>
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Pro Tip: How to remember this:

Translate it into Python in your head!

```python
from numpy.random import randint

def X():
    return randint(1, 7)

def Z():
    return 2 * X()

def D():
    return X() + X()
```

1 random value

2 possibly different random values
Example: Let $X$ = “the number of heads showing when you flip a coin”

What is: $Y = X^2 - X + 3$