Probability in Computing

Lecture 9

CS 237

Reminders

- HW4 due Thursday

Last time

- Conditional Probability

Today

- Conditional Probability
- Tree Diagrams
- Product Rule
- Independent Events

Conditional Probability: Review

Incorporating new information into our probabilistic model

- A test eliminated diseases b and d

	Probability: Before the test		Probability: After the test
Disease a	$\mathbf{0.05}$	→	$\frac{\mathbf{0.05}}{\mathbf{0.35}}$
Disease b	$\mathbf{0.1}$	→	$\mathbf{0}$
Disease c	$\mathbf{0.3}$	→	$\frac{\mathbf{0.3}}{\mathbf{0.35}}$
Disease d	$\mathbf{0.55}$	→	$\mathbf{0}$

Conditional Probability: Review

Definition: Conditional Probability

Let B be an event such that $\Pr(B) \neq 0$. For every event A, we define the conditional probability of event A given event B:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

$\mathbf{Pr}(\boldsymbol{A} \mid \boldsymbol{B})$ is undefined if $\Pr(B) = 0$.

- Conditional probabilities $\Pr(\cdot \mid B)$ satisfy the axioms of probability

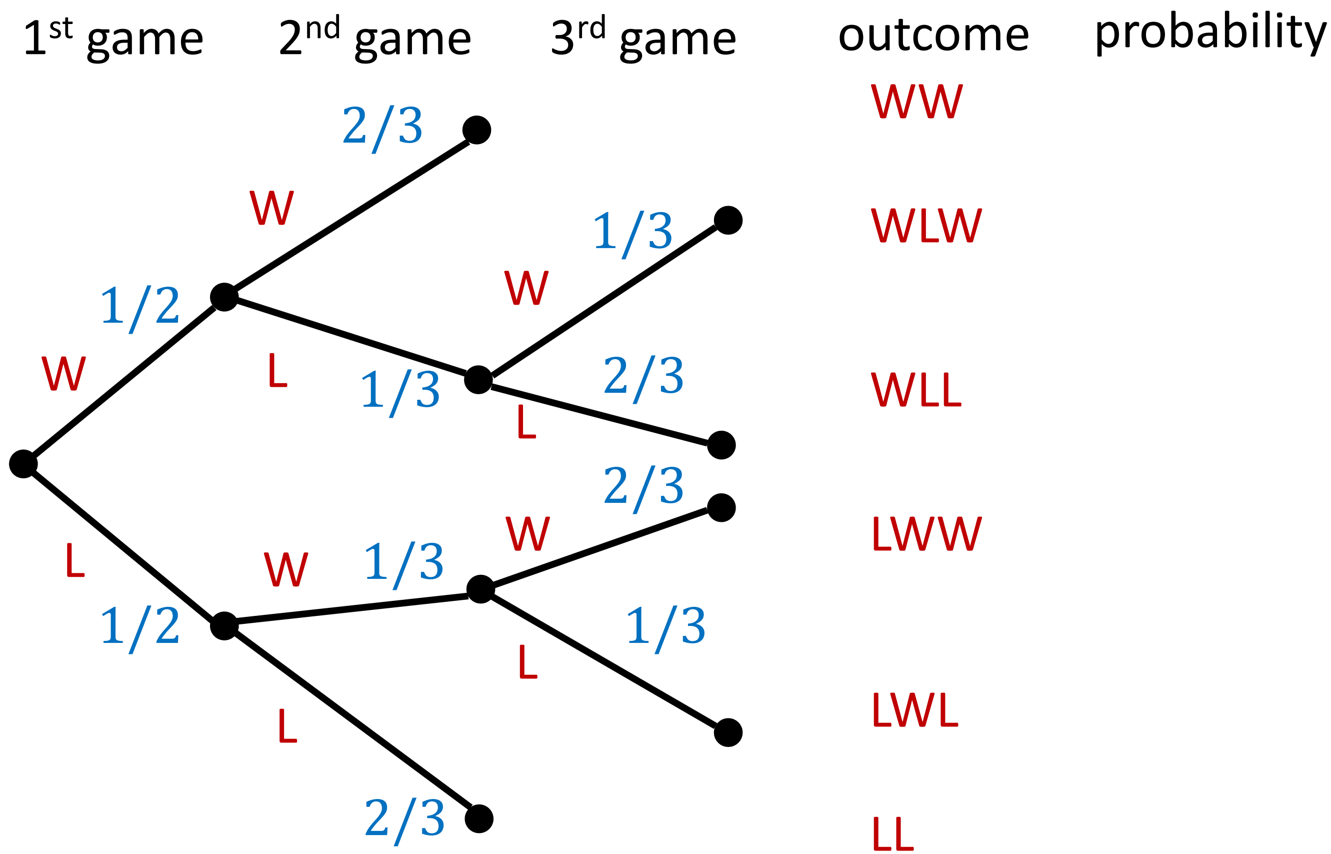
Tree Diagrams: Example

- Best-of-three tournament between BU Terriers and Harvard Huskies
- First game: Terriers win with probability $\frac{1}{2}$
- Subsequent games:

$$\text{Terriers win a game with probability} \begin{cases} \frac{2}{3} & \text{if they won the previous game} \\ \frac{1}{3} & \text{if they lost the previous game} \end{cases}$$

What is the probability that the Terriers win the tournament **given** that they win the first game?

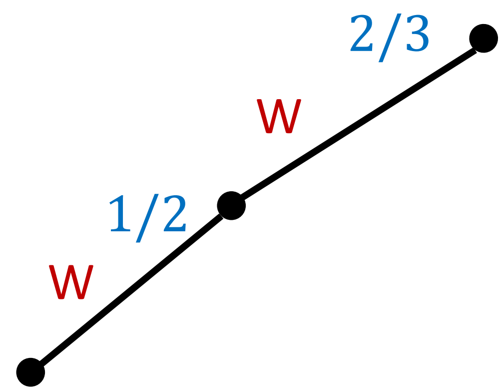
Tree Diagram: Tournament



- Event T =``Terriers win the tournament''
- Event E_1 =``Terriers win the first game''

Tree Diagram: Justification

- Why do we multiply probabilities along each branch in a tree diagram?



- Example:
  - Event E_1 =``Terriers win the first game''
  - Event E_2 =``Terriers win the second game''
- We calculated $\Pr(WW) = \Pr(E_1 \cap E_2)$
$$= \Pr(E_1) \cdot \Pr(E_2 \mid E_1)$$
- Product rule follows from the definition of conditional probability:

$$\Pr(E_2 \mid E_1) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_1)}$$

Product Rule

- The product rule for two events A, B:

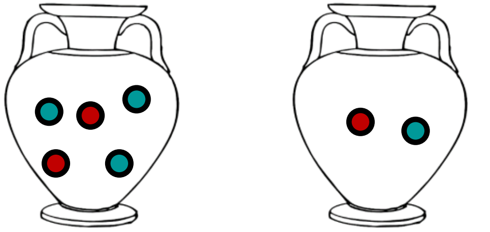
$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(\, B \mid A \,)$$

- It can be generalized to three or more events:

$$\Pr(A \cap B \cap C) = \Pr(A \cap B) \cdot \Pr(\, C \mid A \cap B \,)$$
$$= \Pr(A) \cdot \Pr(\, B \mid A \,) \cdot \Pr(\, C \mid A \cap B \,)$$

Top Hat question (Join Code: 413437)

- We have two urns with colored balls:
  - Urn 1 has 2 red balls and 3 green balls
  - Urn 2 has 1 red balls and 1 green balls

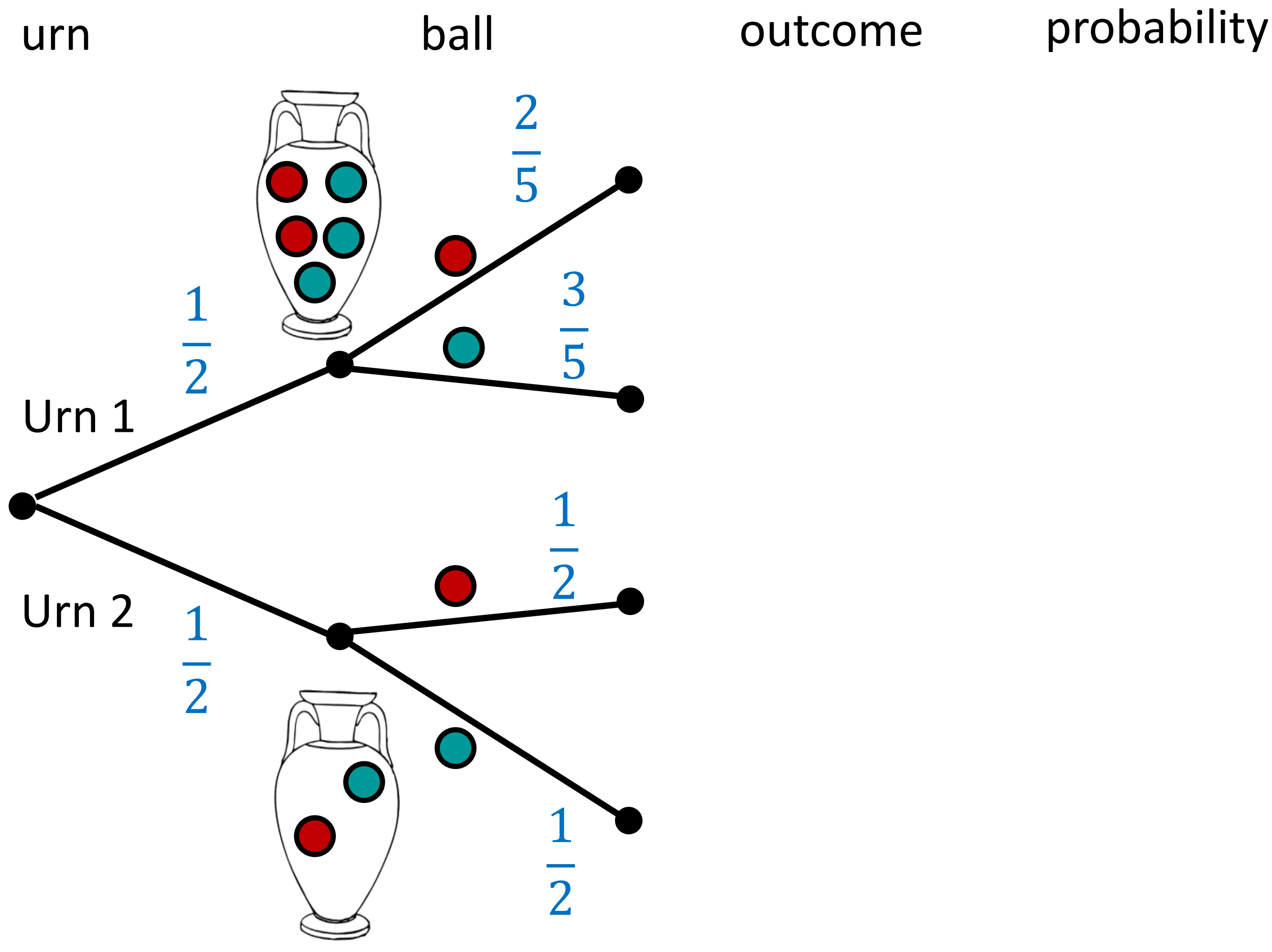


- We pick an urn uniformly at random, and then pick a ball uniformly at random from the chosen urn

Given that the ball we picked is red, what is the probability that it came from Urn 1?

A. 4/9
B. 1/2
C. 2/3
D. None of the above

Tree Diagram Method



Law of Total Probability

- We can use conditional probability to break down probability calculations

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \overline{B})$$
$$= \Pr(A|B) \cdot \Pr(B) + \Pr(A|\overline{B}) \cdot \Pr(\overline{B})$$

by additivity axiom

by product rule

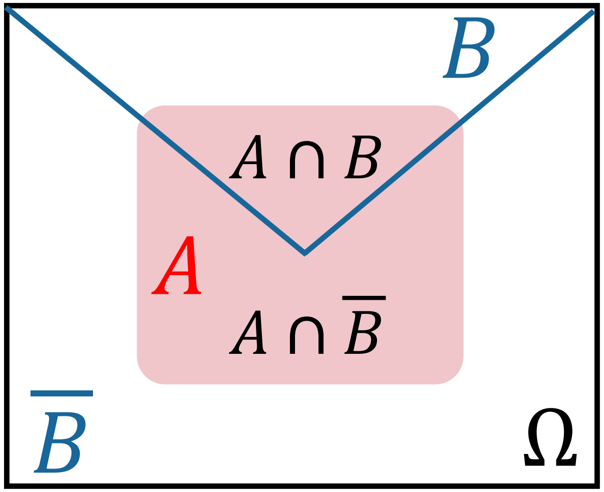
- Law of total probability:

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|\overline{B}) \cdot \Pr(\overline{B})$$

- Example: We have two coins: fair and double-headed

We pick one uniformly at random and toss it.

What is the probability we get heads?



Independent Events

- Conditional probability $\Pr(A \mid B)$ captures partial information that event B provides about event A
- An important setting is when the information that B happened does **not** change the probability of A:

$$\Pr(A \mid B) = \Pr(A)$$

Then, by product rule,

$$\Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B)$$
$$= \Pr(A) \cdot \Pr(B)$$

Independent Events

Definition: Independent Events

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

- Equivalent condition: $\Pr(\, A \mid B \,) = \Pr(A)$

  Alternatively, $\Pr(B \mid A) = \Pr(B)$
- Independence is symmetric:

  if A is independent of B then B is independent of A
- This definition can be used even when $\Pr(B) = 0$
  - An event B with $\Pr(B) = 0$ is independent of all events, including itself.
- **Independent ≠ disjoint!** When events A and B are disjoint: if A happens then B is guaranteed not to happen.

Top Hat question (Join Code: 413437)

Experiment: toss a fair coin twice

- A = “first toss is H”
- B = “second toss is H”

Are *A* and *B* independent?

A. YES

B. NO

Top Hat question (Join Code: 413437)

Experiment: toss two fair coins

- A = “first toss is H”
- B = “both tosses give the same result”

Are *A* and *B* independent?

A. YES

B. NO

Top Hat question (Join Code: 413437)

Experiment: toss two biased coins with Pr(H)= 0.7

- A = "first toss is H"
- B = "both tosses give the same result"

Are *A* and *B* independent?

A. YES

B. NO