Probability in Computing

CS 237

LECTURE 12

Last time

- Bayes’ Rule Example
- Review

Today

- Pairwise and Mutual Independence

Reminders

- HW 6 is due Thursday

Reading

- LLM 18.18-18.19, 19.2
  P 1.4.1, 3.1.4

Midterm Announcement

- Midterm will be Wednesday, March 15, 6:30 – 8:30 in CGS 129
- Closed book, no devices.
- One hand-written 2-sided page of notes allowed on colored paper, significantly different from white. No mechanical reproduction (writing on a computer and then printing is not allowed).
- All material up to tomorrow's discussion (Friday, 3/2)
- Practice exam problems will be distributed soon; Friday discussion will be practice problems, Tuesday lecture (3/14) will be review.
- Review TopHat, homework, and discussion problems, plus exercises suggested from the 2nd textbook.

Reminder: Independent Events

Definition: Independent Events

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

- **Independent ≠ disjoint!** When events A and B are disjoint: if A happens then B is guaranteed not to happen.

Top Hat question (Join Code: 033357)

Let A and B be two **disjoint** events with $\Pr(A) > 0$ and $\Pr(B) > 0$

Are A and B independent?

A. YES, for all A, B satisfying the conditions.

B. NO, for all A, B satisfying the conditions.

C. YES for some A, B satisfying the conditions and NO for others.

Independence for Three Events

How should we define independence for three events A, B, C?

- Options

1) A and B are independent: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$
   B and C are independent: $\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$
   A and C are independent: $\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$
2) $\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$
3) Require both (1) and (2).

Questions: Does (1) imply (2)? Does (2) imply (1)?

Independence for Three Events

Option (1) does not imply Option (2)

- *Experiment:* toss a fair coin twice 
- A = event that 1st toss is H = {HH,HT}
- B = event that 2nd toss is H = {HH,TH}
- C = event that tosses are different = {HT,TH}

Independence for Three Events

Option (2) does not imply Option (1)

- *Experiment:* roll a fair die twice
- A = event that 1st roll is 1, 2 or 3
- B = event that 1st roll is 3, 4 or 5
- C = the sum of the two rolls is 9 = {(3,6),(6,3),(4,5),(5,4)}

$$\Pr(A \cap B \cap C) =$$

$$\Pr(A \cap B) =$$

Independent Events

Definition: Independent Events

Three events A, B, and C are pairwise-independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)\,;$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)\,;$$
$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)\,.$$

Three events A, B and C are mutually independent if they are pairwise independent and

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)\,.$$

Compact Notation: Sum and Product

- Σ: Sigma, summation notation LaTex: \sum
- Π: Pi, product notation LaTex: \prod

$$\sum_{i=2}^{5} i = 2 + 3 + 4 + 5$$

$$\prod_{n=1}^{6} x = x^6$$

$$\prod_{i=1}^{2}\prod_{j=4}^{6} 3ij = \prod_{i=1}^{2}\big((3i \cdot 4)(3i \cdot 5)(3i \cdot 6)\big)$$

$$= (3 \cdot 1 \cdot 4)(3 \cdot 1 \cdot 5)(3 \cdot 1 \cdot 6) \cdot (3 \cdot 2 \cdot 4)(3 \cdot 2 \cdot 5)(3 \cdot 2 \cdot 6)$$

Compact Notation with Sets

- Σ: Sigma, summation notation LaTex: \sum
- Π: Pi, product notation LaTex: \prod

Let $S = \{1,2,3\}$. Then

$$\sum_{i \in S} i^2 =$$

$$\prod_{i \in S} \Pr(A_i) =$$

$$\bigcap_{i \in S} A_i =$$

Independent Events

Definition: Independent Events

Three events A, B, and C are pairwise-independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)\,;$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)\,;$$
$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)\,.$$

Three events A, B and C are mutually independent if they are pairwise independent and

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)\,.$$

- Succinct form of the definition of mutual independence:

Three events A_1, A_2, and A_3 are mutually independent if for all nonempty sets $S \subseteq \{1,2,3\}$,

$$\Pr\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \Pr(A_i)\,.$$

Independence: Examples

1. *Experiment:* toss a fair coin twice
   - $A =$ event that 1st toss is H
   - $B =$ event that 2nd toss is H
   - $C =$ event that tosses are different

   We showed: A, B, C are pairwise independent, but not mutually independent

2. *Experiment:* toss a fair coin 3 times 
   - $A_i =$ event that i-th toss is H for $i = 1,2,3$

Then $\Pr(A_1 \cap A_2) = \Pr(\{HHT, HHH\}) = \frac{1}{4}$

$$= \frac{1}{2} \cdot \frac{1}{2} = \Pr(A_1) \cdot \Pr(A_2)$$

Similarly, for A_1 and A_3 and for A_2 and A_3

$\Pr(A_1 \cap A_2 \cap A_3) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3)$

Conclusion: A_1, A_2, A_3 are mutually independent

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Mutually Independent Events

- We can generalize the definition to any number of events.

Definition: Mutually Independent Events

Events $A_1, A_2, \dots, A_n$ are mutually independent if for all nonempty sets $S \subseteq \{1,2,\dots,n\}$,

$$\Pr\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} \Pr(A_i)\,.$$

Example: toss a fair coin n times

- $A_i =$ event that i-th toss is H for $i = 1,2,\dots,n$

$A_1, A_2, \dots, A_n$ are mutually independent

Caution

- Independence is a very strong assumption
- Many incorrect arguments in probability are due to
  - incorrect assumptions on the independence of events
  - not accounting properly for all of the information available
- Need to make sure we are conditioning on the right events

People v. Collins

Robbery case in 1968 in a suburb of LA

- A purse was snatched from an elderly person
- A couple seen running from the scene were described as a black man with a beard and a mustache and a blond girl with hair in a ponytail
- Witnesses said they drove off in a partly yellow car

Malcolm and Janet Collins were arrested

- He was black and, though clean shaven when arrested, had evidence of recently having had a beard and a mustache
- She was blond and usually wore her hair in a ponytail
- They drove a partly yellow Lincoln

https://en.wikipedia.org/wiki/People_v._Collins

People v. Collins

- The prosecution called a mathematics instructor as a witness, who testified about the multiplication rule for Pr
- The prosecutor suggested to the jury to consider the following odds that the defenders were not the robbers

Black man with beard	1 in 10
Man with mustache	1 in 4
White woman with pony tail	1 in 10
White woman with blond hair	1 in 3
Yellow motor car	1 in 10
Interracial couple in car	1 in 1,000

- What do you think they ruled? (A) Guilty (B) Not guilty
- Do you agree with their decision? (A) Yes (B) No

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People v. Collins

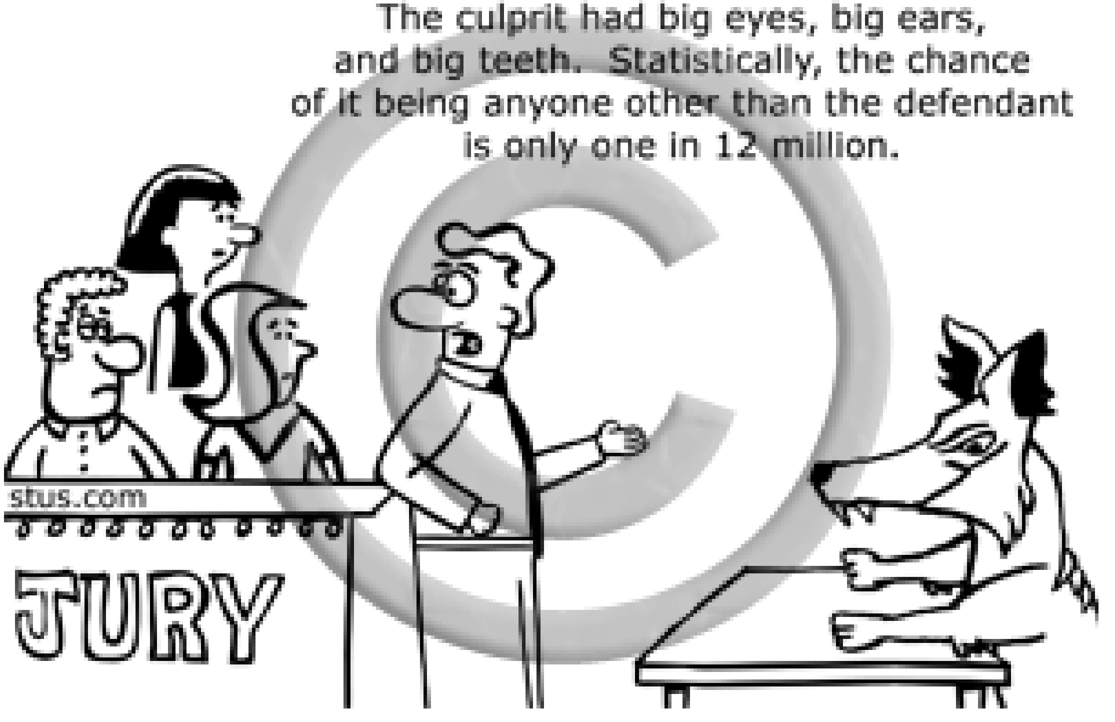
- Modeling assumption: each couple matches the witnesses' description with probability p, independently of the other couples
- Prosecutor's argument:

p = Pr("black man with beard")· Pr("man with mustache") · …

$$= \frac{1}{10} \cdot \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{10} \cdot \frac{1}{1000} = \frac{1}{12{,}000{,}000}$$

Pr("accused couple are not the robbers") = p

People v. Collins



From http://www.stus.com/stus-cartoon.php?name=People+v.+Collins&cartoon=evi0007

People v. Collins: Flaws

- Prosecutor's argument:

$$p = \Pr(\text{“black man with beard”}) \cdot \Pr(\text{“man with mustache”}) \cdot \ldots$$

$$= \frac{1}{10} \cdot \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{10} \cdot \frac{1}{1000} = \frac{1}{12{,}000{,}000}$$

What should this term be by the Product Rule?

- This argument assumes that events are mutually independent

Is it reasonable to assume, e.g., that "black man with beard" and "man with mustache" are independent events?

https://www.pinterest.com/pin/835558537089728385

People v. Collins: Flaws

- Prosecutor's argument:

Pr("accused couple are not the robbers") = Pr("a couple matches the description")

Prosecutor's Fallacy

"The odds of finding this evidence on an innocent man are so small that the jury can safely disregard the possibility that this defendant is innocent"

https://en.wikipedia.org/wiki/Prosecutor's_fallacy

People v. Collins: Flaws

- Prosecutor’s argument:

Pr(“accused couple are not the robbers”) = Pr(“a couple matches the description”)

- This argument fails to condition on all the available information

We need to consider the conditional probability

Pr(“there are at least 2 couples matching description” | “there is at least 1 couple matching description”)

This information was revealed, so we need to condition on it.

People v. Collins

- Suppose there are $\boldsymbol{n}$ couples in the LA area, each matching the description with probability $\boldsymbol{p}$, independently of the others
- Let $X =$ number of couples matching the description

$$\Pr(X \geq 2 \mid X \geq 1) =$$

People v. Collins

- Suppose there are $\boldsymbol{n}$ couples in the LA area, each matching the description with probability $\boldsymbol{p}$, independently of the others
- Let $X =$ number of couples matching the description

$Example$: $n = 5{,}000{,}000$ $\qquad p = \frac{1}{12{,}000{,}000}$

$$\Pr(X \geq 1) = \ 1-(1-p)^n \approx 0.340759$$

$$\Pr(X \geq 2) = \ 1-(1-p)^n - np(1-p)^{n-1} \approx 0.0660758$$

$$\Pr(X \geq 2 \mid X \geq 1) = \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)} \approx 0.193907$$

People v. Collins

- Suppose there are $\boldsymbol{n}$ couples in the LA area, each matching the description with probability $\boldsymbol{p}$, independently of the others
- Let $X =$ number of couples matching the description

Example: $n = 5{,}000{,}000$ $\qquad p = \frac{1}{12{,}000{,}000}$

$$\Pr(X \geq 1) = 1 - (1-p)^n \approx 0.340759$$

$$\Pr(X \geq 2) = 1 - (1-p)^n - np(1-p)^{n-1} \approx \boxed{0.0660758}$$

6.6%: not large, but not small either

$$\Pr(X \geq 2 \mid X \geq 1) = \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)} \approx 0.193907$$

People v. Collins

- Suppose there are $\boldsymbol{n}$ couples in the LA area, each matching the description with probability $\boldsymbol{p}$, independently of the others
- Let $X =$ number of couples matching the description

Example: $n = 5{,}000{,}000$ $\qquad p = \dfrac{1}{12{,}000{,}000}$

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$$\Pr(X \geq 2 \mid X \geq 1) = \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)} \approx \boxed{0.193907}$$

19.39%: that's pretty high

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People v. Collins

- The California Supreme Court overruled the initial guilty verdict
- There are at least two flaws in the prosecution's argument:
  - Independence assumptions: "black man with beard" and "man with mustache" may not be independent
  - Not conditioning on all the available information ("at least one such couple")