Probability in Computing

LECTURE 13

CS 237

Reminders

- HW 6 is due Thursday

Reading

- LLM 18.18-18.19, 19.2
  P 1.4.1, 3.1.4

Last time

- Independent Events
- Sequences of Independent Events
- Bayes' Rule

Today

- Independent Random Variables
- Pairwise and Mutual Independence

Reminder: Independent Events

Definition: Independent Events

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

- **Independent ≠ disjoint!** When events A and B are disjoint: if A happens then B is guaranteed not to happen.

Top Hat question (Join Code: 413437)

You have two fair coins: gold and silver.

Experiment: toss each coin 3 times

- G = "gold coin came up HEADS all 3 times"
- S = "silver coin came up HEADS all 3 times"

1. Are *G* and *S* independent?
2. Are *G* and *S* disjoint?

A. YES to both

B. NO to both

C. 1. YES 2. NO

D. 1. NO 2. YES

Top Hat question (Join Code: 413437)

You have a fair 4-sided die and two fair coins: gold and silver.

Experiment: roll the die, let D be the number rolled,
and toss each coin D times

- G = "gold coin came up HEADS all *D* times"
- S = "silver coin came up HEADS all *D* times"

1. Are *G* and *S* independent?
2. Are *G* and *S* disjoint?

A. YES to both
B. NO to both
C. 1. YES 2. NO
D. 1. NO 2. YES

Independent Random Variables

Definition: Independent Random Variables

Random variables X and Y are independent if **for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$,** the events $[X \leq x]$ and $[Y \leq y]$ are independent, i.e.,
$$\Pr([X \leq x] \wedge [Y \leq y]) = \Pr(X \leq x) \cdot \Pr(Y \leq y).$$

- This definition applies to both discrete and continuous RVs.
- For discrete random variables, it is equivalent to:

Definition: Discrete Independent Random Variables

Discrete random variables X and Y are independent if

for all $\boldsymbol{x} \in \mathrm{range}(\boldsymbol{X}), \boldsymbol{y} \in \mathrm{range}(\boldsymbol{Y})$,

the events $[X = x]$ and $[Y = y]$ are independent.

Top Hat question (Join Code: 413437)

Experiment: toss a fair coin 3 times

- C = number of heads
- $M = \begin{cases} 1 & \text{if all three outcomes match (are the same)} \\ 0 & \text{otherwise} \end{cases}$

Are C and M independent?

A. YES

B. NO

Top Hat question (Join Code: 413437)

Experiment: toss a fair coin 3 times

- $H_1 = \begin{cases} 1 \text{ if the first toss is } H \\ 0 \text{ otherwise} \end{cases}$
- $M = \begin{cases} 1 \text{ if all three outcomes match} \\ 0 \text{ otherwise} \end{cases}$

Are H_1 and M independent?

A. YES

B. NO

Top Hat question

Experiment: toss a fair coin 3 times

- $H_1 = \begin{cases} 1 \text{ if the first toss is } H \\ 0 \text{ otherwise} \end{cases}$
- $M = \begin{cases} 1 \text{ if all three outcomes match} \\ 0 \text{ otherwise} \end{cases}$

Are H_1 and M independent?

	$\boldsymbol{M = 1}$	$\boldsymbol{M = 0}$
Pr	1/4	3/4

	Pr
$\boldsymbol{H_1 = 1}$	1/2
$\boldsymbol{H_1 = 0}$	1/2

	$\boldsymbol{M = 1}$	$\boldsymbol{M = 0}$
$\boldsymbol{H_1 = 1}$	1/8	3/8
$\boldsymbol{H_1 = 0}$	1/8	3/8

Independence for Three Events

How should we define independence for three events A, B, C?

- Options

1) A and B are independent: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$
   B and C are independent: $\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$
   A and C are independent: $\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$
2) $\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$
3) Require both (1) and (2).

Questions: Does (1) imply (2)? Does (2) imply (1)?

Independence for Three Events

Option (1) does not imply Option (2)

- *Experiment:* toss a fair coin twice 
- A = event that 1st toss is H = {HH,HT}
- B = event that 2nd toss is H = {HH,TH}
- C = event that tosses are different = {HT,TH}

Independence for Three Events

Option (2) does not imply Option (1)

- *Experiment:* roll a fair die twice
- A = event that 1st roll is 1, 2 or 3
- B = event that 1st roll is 3, 4 or 6
- C = the sum of the two rolls is 9 = {(3,6),(6,3),(4,5),(5,4)}

$$\Pr(A \cap B \cap C) =$$

$$\Pr(A \cap B) =$$