Probability in Computing

CS 237

LECTURE 14

Last time (before review)

- Pairwise and Mutual Independence

Today

- Independence of Random Variables

Reminders

- HW 7 is due next Thursday

Reading

- LLM 18.18-18.19, 19.2
  P 1.4.1, 3.1.4

Reminder: Independent Events

Definition: Independent Events

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

Definition: Three Independent Events

Three events $A, B,$ and C are pairwise-independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)\,;$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)\,;$$
$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)\,.$$

Three events A, B and C are mutually independent if they are pairwise independent and

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)\,.$$

Mutually Independent Events

- We can generalize the definition to any number of events.

Definition: Mutually Independent Events

Events $A_1, A_2, \dots, A_n$ are mutually independent if for all nonempty sets $S \subseteq \{1,2,\dots,n\}$,

$$\Pr\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} \Pr(A_i)\,.$$

Example: toss a fair coin n times

- $A_i =$ event that i-th toss is H for $i = 1,2,\dots,n$

$A_1, A_2, \dots, A_n$ are mutually independent

Top Hat question (Join Code: 033357)

How many (nontrivial) equalities do we have to check to establish mutual independence of n events?

A. n
B. $3n-5$
C. $(n-1)^2$
D. 2^n-n-1
E. None of the above

Top Hat question (Join Code: 033357)

We roll a 6-sided die n times.

- Let A_i be the event that the i-th roll is 6 for all $i = 1, \ldots, n$
- Let $P_{i,j} = A_i \cap A_j$ for all i and $j \in \{1, \ldots, n\}$ where $i < j$
- Let $E_{i,j}$ be the event that the results of rolls i and j are the same for all i and $j \in \{1, \ldots, n\}$ where $i < j$

Which of the following events are mutually independent?

(Pick the last option that applies)

A. $A_1, A_2, \ldots, A_n$ if the die is fair

B. $A_1, A_2, \ldots, A_n$ if the die is fair or unfair (it does not matter)

C. $P_{1,2}, P_{3,4}, P_{5,6}$

D. All events $E_{i,j}$ (that is, $E_{1,2}, E_{1,3}, \ldots, E_{n-1,n}$) when the die is fair

E. All events $P_{i,j}$ (that is, $P_{1,2}, P_{1,3}, \ldots, P_{n-1,n}$) when the die is fair

Independent Random Variables

Definition: Independent Random Variables

Random variables X and Y are independent if **for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$,** the events $[X \leq x]$ and $[Y \leq y]$ are independent, i.e.,

$$\Pr([X \leq x] \wedge [Y \leq y]) = \Pr(X \leq x) \cdot \Pr(Y \leq y).$$

- This definition applies to both discrete and continuous RVs.
- For discrete random variables, it is equivalent to:

Definition: Discrete Independent Random Variables

Discrete random variables X and Y are independent if **for all $\boldsymbol{x} \in \text{range}(\boldsymbol{X})$** and **$\boldsymbol{y} \in \text{range}(\boldsymbol{Y})$,** the events $[X = x]$ and $[Y = y]$ are independent.

Examples: 1) X and Y are the results of two rolls of a die.

2) X and Y the are distances of two darts from the center of the target.

Top Hat question (Join Code: 033357)

Experiment: toss a fair coin 3 times

- C = number of heads
- $M = \begin{cases} 1 & \text{if all three outcomes match (are the same)} \\ 0 & \text{otherwise} \end{cases}$

Are C and M independent?

A. YES

B. NO

Top Hat question (Join Code: 033357)

Experiment: toss a fair coin 3 times

- $H_1 = \begin{cases} 1 \text{ if the first toss is } H \\ 0 \text{ otherwise} \end{cases}$
- $M = \begin{cases} 1 \text{ if all three outcomes match} \\ 0 \text{ otherwise} \end{cases}$

Are H_1 and M independent?

A. YES

B. NO

Top Hat question

Experiment: toss a fair coin 3 times

- $H_1 = \begin{cases} 1 \text{ if the first toss is } H \\ 0 \text{ otherwise} \end{cases}$
- $M = \begin{cases} 1 \text{ if all three outcomes match} \\ 0 \text{ otherwise} \end{cases}$

Are H_1 and M independent?

	$\boldsymbol{M = 1}$	$\boldsymbol{M = 0}$
Pr	1/4	3/4

	Pr
$\boldsymbol{H_1 = 1}$	1/2
$\boldsymbol{H_1 = 0}$	1/2

	$\boldsymbol{M = 1}$	$\boldsymbol{M = 0}$
$\boldsymbol{H_1 = 1}$	1/8	3/8
$\boldsymbol{H_1 = 0}$	1/8	3/8