Lecture 15

Last time
• Pairwise and Mutual Independence

Today
• Finish independence of random variables
• Expected value of a random variable (can be interpreted as the average value of a random variable)

Reminders
• HW 7 is due Thursday

Reading
• LLM 19.4-19.5
• P 3.2.2
Definition: Independent Random Variables

Random variables $X$ and $Y$ are independent if for all $x, y \in \mathbb{R}$, the events $[X \leq x]$ and $[Y \leq y]$ are independent, i.e.,
$$\Pr([X \leq x] \land [Y \leq y]) = \Pr(X \leq x) \cdot \Pr(Y \leq y).$$

- This definition applies to both discrete and continuous RVs.
- For discrete random variables, it is equivalent to:

Definition: Discrete Independent Random Variables

Discrete random variables $X$ and $Y$ are independent if for all $x \in \text{range}(X)$ and $y \in \text{range}(Y)$,
the events $[X = x]$ and $[Y = y]$ are independent.

Examples: 1) $X$ and $Y$ are the results of two rolls of a die.
2) $X$ and $Y$ the are distances of two darts from the center of the target.
Review Exercise

- Let $X$ and $Y$ be **independent** random variables, each taking on the values -1 and 1 with probability 1/2.
- Let $Z = X \cdot Y$.

Find the PMF of $Z$
• Let $X$ and $Y$ be independent random variables, each taking on the values -1 and 1 with probability $1/2$.

• Let $Z = X \cdot Y$.

Are $X, Y$, and $Z$ pairwise independent?

A. YES

B. NO
Mutually Independent Random Variables

- Definition of mutual independence carries over from events to RVs

**Definition: Mutually Independent RVs**

Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if for all values $x_1, \ldots, x_n \in \mathbb{R}$, the events $[X_1 \leq x_1], [X_2 \leq x_2], \ldots, [X_n \leq x_n]$ are mutually independent.

- For discrete random variables, we can replace the events with $[X_1 = x_1], [X_2 = x_2], \ldots, [X_n = x_n]$

Examples: 1) $X_1, \ldots, X_n$ are the results of $n$ rolls of a die.
2) $X_1, \ldots, X_n$ are the distances of $n$ darts from the center of the target.
Let $X$ and $Y$ be independent random variables, each taking on the values -1 and 1 with probability 1/2.

Let $Z = X \cdot Y$.

Are $X$, $Y$, and $Z$ mutually independent?

A. YES

B. NO
Expectation of Random Variables
You roll one die. 🎲
Let $X$ be the random variable that represents the result.

What value does $X$ take on average?

A. $1/6$
B. $3$
C. $3.5$
D. $6$
E. None of the above.
Example: Spinner

- Spin the dial of the spinner.
  Let \( Y \) be the number of the region where it stopped.
  - Range(\( Y \)) = \{1,2,3,4\}
  - \( \Pr(Y = 1) = \frac{1}{2}, \Pr(Y = 2) = \frac{1}{4}, \Pr(Y = 3) = \Pr(Y = 4) = \frac{1}{8} \)
- What value does \( Y \) take on average?
  - It is NOT \( \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \)
- Suppose we spin the dial \( N \) times, where \( N \) is huge. Then we expect to see about \( \frac{N}{2} \) ones, \( \frac{N}{4} \) twos, \( \frac{N}{8} \) threes, and \( \frac{N}{8} \) fours.
- If we add them up and divide by \( N \), we get
  \[
  \frac{\frac{N}{2} \cdot 1 + \frac{N}{4} \cdot 2 + \frac{N}{8} \cdot 3 + \frac{N}{8} \cdot 4}{N} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{2} = \frac{15}{8} = 1.875
  \]
  \( \neq 2.5 \)

We want a **weighted average**: each value is counted the number of times proportional to its probability.
Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable $X$ over a sample space $\Omega$ is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

- **Example 1:** $X =$ number obtained when rolling a die
  $$\mathbb{E}(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

- **Example 2:** $Y =$ region of the spinner selected
  $$\mathbb{E}(Y) = \sum_{i=1}^{4} i \cdot \Pr(i) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{8} = \frac{15}{8}$$

Expectation of $X$ does not have to be in Range($X$)
Example: roulette

- 38 slots: 18 black, 18 red, 2 green.

- If we bet $1 on red, we get $2 back if red comes up. What’s the expected value of our winnings?

- Let $X$ be the value of winnings

  It is tedious to consider each slot separately

  We can combine outcomes on which $X$ takes the same value
Random variables: expectation

### Definition: Expectation

The **expectation** (also called the **expected value** or **mean**) of a discrete random variable $X$ over a sample space $\Omega$ is

$$
\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).
$$

- We can group together outcomes $\omega$ for which $X(\omega) = a$:

$$
\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a).
$$

**Proof:**

$$
\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)
$$

$$
= \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega : X(\omega) = a} X(\omega) \cdot \Pr(\omega)
$$

$$
= \sum_{a \in \text{Range}(X)} a \cdot \sum_{\omega \in \Omega : X(\omega) = a} \Pr(\omega)
$$

$$
= \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)
$$
Example: roulette

- 38 slots: 18 black, 18 red, 2 green.
- If we bet $1 on red, we get $2 back if red comes up. What’s the expected value of our winnings?
- Let $X$ be the value of winnings

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)$$
You roll two dice. Let $X_1$ be the number on the 1st die, $X_2$ be the number on the 2nd die, and $X = |X_1 - X_2|$. Find $E(X)$.

A. 0  
B. 1  
C. $\frac{70}{36}$  
D. 2  
E. 3.5  
F. None of the above