Probability in Computing

CS 237

LECTURE 15

Last time

- Pairwise and Mutual Independence

Today

- Expected value of a random variable
  (can be interpreted as the average value of a random variable)

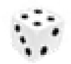
Reminders

- HW 7 is due Thursday

Reading

- LLM 19.4-19.5
  P 3.2.2

Top Hat question (Join Code: 413437)

You roll one die. 

Let X be the random variable that represents the result.

What value does X take on average?

A. 1/6

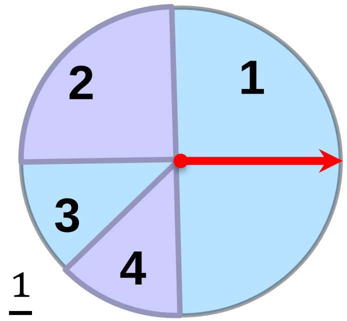
B. 3

C. 3.5

D. 6

E. None of the above.

Example: Spinner



- Spin the dial of the spinner.
  Let Y be the number of the region where it stopped.
  - $\text{Range}(Y) = \{1,2,3,4\}$
  - $\Pr(Y = 1) = \frac{1}{2},\ \Pr(Y = 2) = \frac{1}{4}, \Pr(Y = 3) = \Pr(Y = 4) = \frac{1}{8}$
- What value does Y take on average?
  - It is NOT $\frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$
- Suppose we spin the dial N times, where N is huge. Then we expect to see about $\frac{N}{2}$ ones, $\frac{N}{4}$ twos, $\frac{N}{8}$ threes, and $\frac{N}{8}$ fours.
- If we add them up and divide by N, we get

$$\frac{\frac{N}{2} \times 1 + \frac{N}{4} \times 2 + \frac{N}{8} \times 3 + \frac{N}{8} \times 4}{N} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{2} = \frac{15}{8} = 1.875 \quad \boxed{\neq 2.5}$$

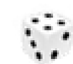
We want a *weighted average*:
each value is counted the number of times proportional to its probability.

Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable X over a sample space Ω is

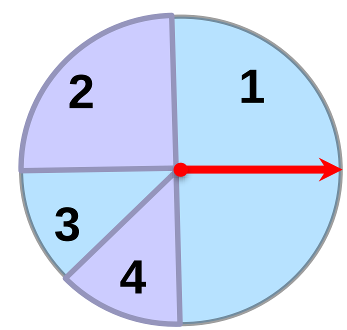
$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

- Example 1: $X =$ number obtained when rolling a die 

$$\mathbb{E}(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

- Example 2: $Y =$ region of the spinner selected

$$\mathbb{E}(Y) = \sum_{i=1}^{4} i \cdot \Pr(i) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{8} = \frac{15}{8}$$



Expectation of X does not have to be in Range(X)

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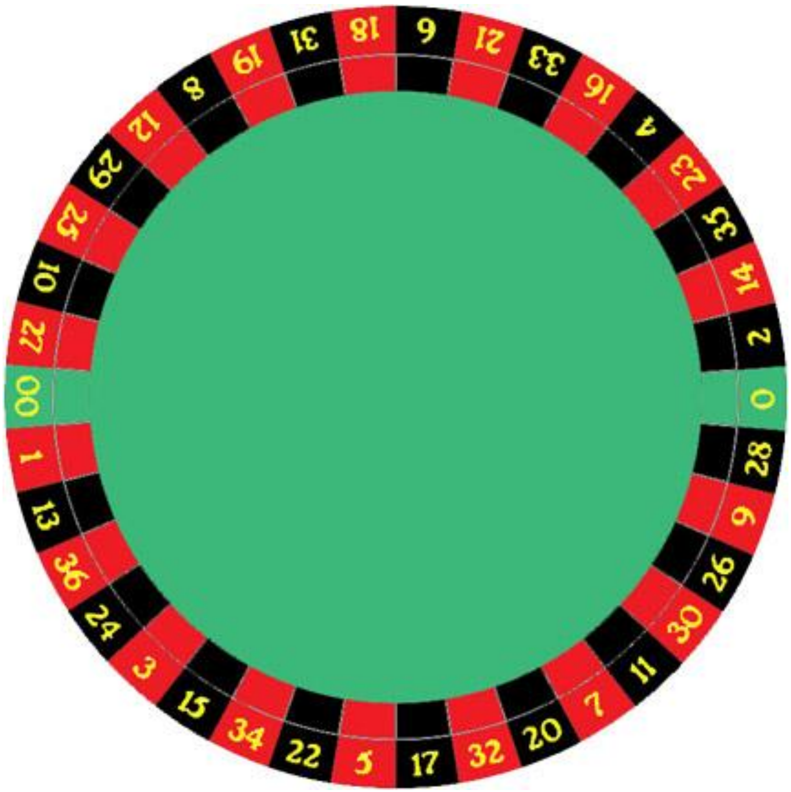
Example: roulette

- 38 slots: 18 black, 18 red, 2 green.



- If we bet \$1 on red, we get \$2 back if red comes up. What's the expected value of our winnings?
- Let X be the value of winnings

It is tedious to consider each slot separately

We can combine outcomes on which X takes the same value

Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable X over a sample space Ω is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

- We can group together outcomes ω for which $X(\omega) = a$:

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a).$$

This version of the definition is more useful for computations.

Proof: $\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$

$$= \sum_{\boldsymbol{a \in \text{Range}(X)}} \sum_{\boldsymbol{\omega \in \Omega : X(\omega) = a}} X(\omega) \cdot \Pr(\omega) = \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega : X(\omega) = a} \boldsymbol{a} \cdot \Pr(\omega)$$

$$= \sum_{a \in \text{Range}(X)} \boldsymbol{a} \cdot \sum_{\omega \in \Omega : X(\omega) = a} \Pr(\omega) = \sum_{a \in \text{Range}(X)} a \cdot \boldsymbol{\Pr(X = a)}$$

Example: roulette

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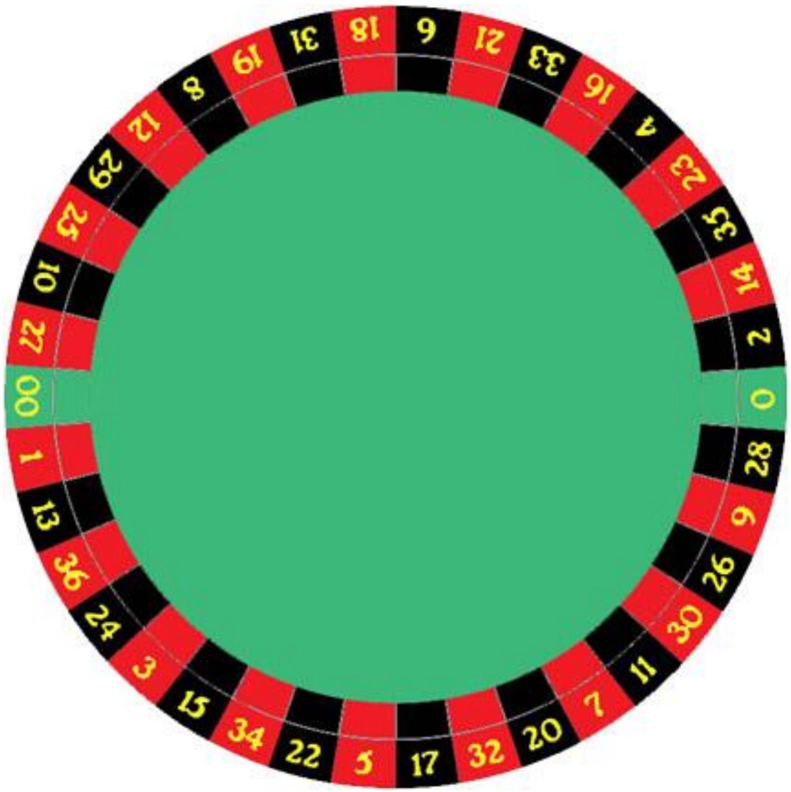

- If we bet \$1 on red, we get \$2 back if red comes up. What's the expected value of our winnings?
- Let X be the value of winnings

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)$$

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Top Hat question (Join Code: 413437)

You roll two dice. 

Let X_1 be the number of the 1st die, X_2 be the number on the 2nd die, and $X = |X_1 - X_2|$. Find $\mathbb{E}(X)$.

A. 0

B. 1

C. $\frac{70}{36}$

D. 2

E. 3.5

F. None of the above

Top Hat question (Join Code: 413437)

You toss a fair coin until you get heads. 

Let X be the number of tosses. What is $\mathbb{E}(X)$?

A. 1.5

B. $\sqrt{2}$

C. 2

D. 4

E. Infinite

Linearity of Expectation

Theorem. For any two random variables X and Y on the same probability space,

$$\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y).$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}(cX)=c\cdot\mathbb{E}(X).$$

- You roll two dice.

Let X_1 be the number of the 1st die, X_2 be the number on the 2nd die.

Example 1: Find $\mathbb{E}(X_1+X_2)$.

Example 2: Find $\mathbb{E}(2X_1)$.

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Linearity of Expectation

Theorem. For any two random variables X and Y on the same probability space,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}(X) = c \cdot \mathbb{E}(X).$$

- More generally, for all $a, b \in \mathbb{R}$,

$$\mathbb{E}(aX + bY) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y).$$

- And for multiple random variables $X_1, \dots, X_n$ and numbers $c_1, \dots, c_n \in \mathbb{R}$,

$$\mathbb{E}(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n) \\ = c_1 \mathbb{E}(X_1) + c_2 \mathbb{E}(X_2) + \cdots + c_n \mathbb{E}(X_n)$$

Linearity of expectation holds even for dependent random variables!