LECTURE 15

Last time
• Pairwise and Mutual Independence

Today
• Expected value of a random variable
  (can be interpreted as the average value of a random variable)

Reminders
• HW 7 is due Thursday

Reading
• LLM 19.4-19.5
  P 3.2.2
You roll one die. 🎲
Let X be the random variable that represents the result.

What value does X take on average?

A. $\frac{1}{6}$
B. 3
C. 3.5
D. 6
E. None of the above.
Example: Spinner

- Spin the dial of the spinner.
  Let $Y$ be the number of the region where it stopped.
  - Range($Y$) = \{1,2,3,4\}
  - $\Pr(Y = 1) = \frac{1}{2}$, $\Pr(Y = 2) = \frac{1}{4}$, $\Pr(Y = 3) = \Pr(Y = 4) = \frac{1}{8}$

- What value does $Y$ take on average?
  - It is NOT $\frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$

- Suppose we spin the dial $N$ times, where $N$ is huge. Then we expect to see about $\frac{N}{2}$ ones, $\frac{N}{4}$ twos, $\frac{N}{8}$ threes, and $\frac{N}{8}$ fours.

- If we add them up and divide by $N$, we get
  \[ \frac{\frac{N}{2} \times 1 + \frac{N}{4} \times 2 + \frac{N}{8} \times 3 + \frac{N}{8} \times 4}{N} = \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{2} = \frac{15}{8} \approx 1.875 \neq 2.5 \]

We want a **weighted average**: each value is counted the number of times proportional to its probability.
Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable $X$ over a sample space $\Omega$ is

$$
\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \text{Pr}(\omega).
$$

- **Example 1:** $X =$ number obtained when rolling a die
  $$
  \mathbb{E}(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{21}{6} = 3.5
  $$

- **Example 2:** $Y =$ region of the spinner selected
  $$
  \mathbb{E}(Y) = \sum_{i=1}^{4} i \cdot \text{Pr}(i) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{8} = \frac{15}{8}
  $$

Expectation of $X$ does not have to be in Range($X$)
Example: roulette

- 38 slots: 18 black, 18 red, 2 green.

- If we bet $1 on red, we get $2 back if red comes up. What’s the expected value of our winnings?

- Let $X$ be the value of winnings

It is tedious to consider each slot separately

We can combine outcomes on which $X$ takes the same value
Random variables: expectation

**Definition: Expectation**

The expectation (also called the expected value or mean) of a discrete random variable $X$ over a sample space $\Omega$ is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

- We can group together outcomes $\omega$ for which $X(\omega) = a$:

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a).$$

**Proof:**

\[
\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega) \\
= \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega: X(\omega) = a} X(\omega) \cdot \Pr(\omega) \\
= \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega: X(\omega) = a} a \cdot \Pr(\omega) \\
= \sum_{a \in \text{Range}(X)} a \cdot \sum_{\omega \in \Omega: X(\omega) = a} \Pr(\omega) \\
= \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)
\]
Example: roulette

- 38 slots: 18 black, 18 red, 2 green.

- If we bet $1 on red, we get $2 back if red comes up. What’s the expected value of our winnings?

- Let $X$ be the value of winnings

$$
\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)
$$
You roll two dice.

Let $X_1$ be the number of the 1st die, $X_2$ be the number on the 2nd die, and $X = |X_1 - X_2|$. Find $E(X)$.

A. 0  
B. 1  
C. $\frac{70}{36}$  
D. 2  
E. 3.5  
F. None of the above
You toss a fair coin until you get heads. Let $X$ be the number of tosses. What is $E(X)$?

A. 1.5
B. $\sqrt{2}$
C. 2
D. 4
E. Infinite
Linearity of Expectation

**Theorem.** For any two random variables $X$ and $Y$ on the same probability space,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}(cX) = c \cdot \mathbb{E}(X).$$

- You roll two dice.
  Let $X_1$ be the number of the 1st die, $X_2$ be the number on the 2nd die.

  **Example 1:** Find $\mathbb{E}(X_1 + X_2)$.

  **Example 2:** Find $\mathbb{E}(2X_1)$. 

Linearity of Expectation

**Theorem.** For any two random variables $X$ and $Y$ on the same probability space,

$$
\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).
$$

Also, for all $c \in \mathbb{R}$,

$$
\mathbb{E}(X) = c \cdot \mathbb{E}(X).
$$

- More generally, for all $a, b \in \mathbb{R},$

  $$
  \mathbb{E}(aX + bY) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y).
  $$

- And for multiple random variables $X_1, \ldots, X_n$ and numbers $c_1, \ldots, c_n \in \mathbb{R},$

  $$
  \mathbb{E}(c_1X_1 + c_2X_2 + \cdots + c_nX_n) \\
  = c_1 \mathbb{E}(X_1) + c_2 \mathbb{E}(X_2) + \cdots + c_n \mathbb{E}(X_n)
  $$

Linearity of expectation holds even for dependent random variables!