LECTURE 16

Last time
• Finish independence of random variables
• Expected value of a random variable

Today
• Expectation and infinite sums
• Linearity of expectation

Reminders
• HW 7 is due Thursday

Reading
• LLM 19.4-19.5
  P 3.2.2
Reminder: expectation

**Definition: Expectation**

The expectation (also called the expected value or mean) of a discrete random variable $X$ over a sample space $\Omega$ is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

This version of the definition is more useful for computations.

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a).$$
You toss a fair coin until you get heads. Let $X$ be the number of tosses. What is $E(X)$?

A. 1.5
B. $\sqrt{2}$
C. 2
D. 4
E. Infinite
Sum of Geometric Series: Review

- Find $\sum_{k=1}^{\infty} \frac{1}{2^k} =$

- Find $\sum_{k=0}^{\infty} ar^k =$
You toss a fair coin until you get heads. Let $X$ be the number of tosses. What is $\mathbb{E}(X)$?

$$\mathbb{E}(X) = \sum_{a=1}^{\infty} a \cdot \Pr(X = a) =$$

$$\Pr(X = a) = \Pr\left(\{T \ldots T \, H\}\right) =$$

$a = 1$: $\frac{1}{2}$

$a = 2$: $\frac{1}{2^2} + \frac{1}{2^2}$

$a = 3$: $\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$
Solution 2: Verifying Rearrangement

\[
\sum_{a=1}^{\infty} \Pr(X \geq a) = \sum_{a=1}^{\infty} a \cdot \Pr(X = a)
\]

\[\Pr(X \geq 1) = \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \cdots\]

\[\Pr(X \geq 2) = \]

\[\Pr(X \geq 3) = \]

\[\vdots \]

\[\sum_{a=1}^{\infty} \Pr(X \geq a) = \]
Theorem. For any two random variables $X$ and $Y$ on the same probability space,
\[ \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y). \]

Also, for all $c \in \mathbb{R}$,
\[ \mathbb{E}(cX) = c \cdot \mathbb{E}(X). \]

• You roll two dice.

Let $X_1$ be the number of the 1st die, $X_2$ be the number on the 2nd die.

Example 1: Find $\mathbb{E}(X_1 + X_2)$.

Example 2: Find $\mathbb{E}(2X_1)$. 
Linearity of Expectation

**Theorem.** For any two random variables $X$ and $Y$ on the same probability space,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}(cX) = c \cdot \mathbb{E}(X).$$

- More generally, for all $a, b \in \mathbb{R}$,
  $$\mathbb{E}(aX + bY) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y).$$

- And for multiple random variables $X_1, \ldots, X_n$ and numbers $c_1, \ldots, c_n \in \mathbb{R}$,
  $$\mathbb{E}(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \cdots + c_n\mathbb{E}(X_n)$$

Linearity of expectation holds even for dependent random variables!
An indicator random variable takes on two values: 0 and 1.

Lemma. For an indicator random variable $Y$, 
$$E(Y) = \Pr(Y = 1).$$

Proof:

An indicator random variable $Y$ for an event $E$ is
$$Y = \begin{cases} 
1 & \text{if } E \text{ occurs} \\
0 & \text{otherwise}
\end{cases}$$

Method for computing expectation of a random variable $X$: Represent $X$ as a sum of indicators and use linearity of expectation.
You have a coin that shows HEADS with probability $3/4$. Let $X$ be the number of HEADS in 1000 tosses of your coin. You represent $X$ as the sum: $X = X_1 + X_2 + \cdots + X_{1000}$.

What is $X_1$?


B. The number of HEADS.

C. The probability of HEADS in toss 1.

D. The number of heads in toss 1.

E. None of the above.
You have a coin that shows HEADS with probability $\frac{3}{4}$. Let $X$ be the number of HEADS in 1000 tosses of your coin.

What is the expectation of $X$?

A. $\frac{3}{4}$.
B. $\frac{4}{3}$.
C. 500.
D. 750.
E. None of the above.
Example: Random Proofs

• **Infinite Monkey Theorem.** Given enough time, a hypothetical chimpanzee typing at random would, as part of its output, almost surely produce all of Shakespeare’s plays (or any other text.)

  [Link](https://en.wikipedia.org/wiki/Infinite_monkey_theorem)

• Suppose that the monkey types on a 26-letter keyboard

• Each letter is chosen independently and uniformly at random from the 26 letters

• Suppose the monkey types $n = 1,000,000,000$ letters

• Let $X =$ number of times the sequence “proof” appears

• What is $E(X)$?
Example: Random Proofs

• Suppose the monkey types $n = 1,000,000,000$ letters
• Let $X =$ number of times the sequence “proof” appears
• What is $\mathbb{E}(X)$?

Solution: For each position $i$ in the sequence, let