Probability in Computing

CS 237

Lecture 16

Last time

- Expected value of a random variable

Today

- Expectation of continuous random variables
- Linearity of expectation

Reminders

- HW 7 is due Thursday

Reading

- LLM 19.4-19.5
  P 3.2.2

Reminder: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable X over a sample space Ω is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

$$\mathbb{E}(X) = \sum_{a \in \mathrm{Range}(X)} a \cdot \Pr(X = a).$$

This version of the definition is more useful for computations.

Expectation for Continuous RVs

- The expectation of a continuous random variable is defined as in the discrete case, but we replace the sum with an integral
- Let X be a random variable with PDF f

discrete:

$$\mathbb{E}(X) = \sum_{x \in \mathbb{R}} x \cdot f(x)$$

continuous:

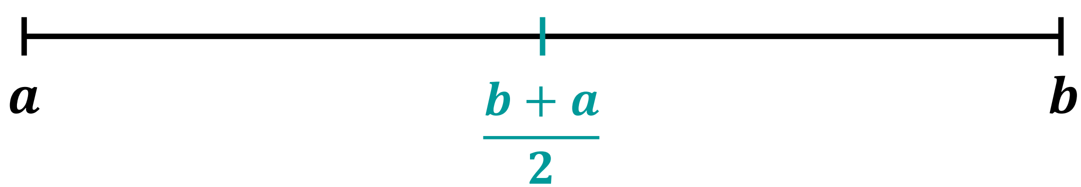
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Example: Uniform Random Variable

- Suppose X is chosen uniformly at random from the interval $[a, b]$
- The PDF is $f(x) = \frac{1}{b-a}$ for $x \in [a, b]$
- $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x\, dx$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{b+a}{2}$$

- Note: $\mathbb{E}(X)$ is the midpoint of the interval $[a, b]$



Example: Darts

- Suppose we throw a dart at a circular board of radius r
- Let D be the distance from the center of the board to the hitting point of the dart
- The PDF is $f(x) = \frac{2x}{r^2}$ for $x \in [0, r]$
- $\mathbb{E}(D) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx \quad = \int_0^r x \cdot \frac{2x}{r^2} dx \quad = \frac{2}{r^2} \int_0^r x^2\,dx$

$$= \frac{2}{r^2} \left. \frac{x^3}{3} \right|_0^r \quad = \frac{2}{r^2} \frac{r^3}{3} \quad = \frac{2r}{3}$$

Linearity of Expectation

Theorem. For any two random variables X and Y on the same probability space,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}(cX) = c \cdot \mathbb{E}(X).$$

- More generally, for all $a, b \in \mathbb{R}$,

$$\mathbb{E}(aX + bY) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y).$$

- And for multiple random variables $X_1, \ldots, X_n$ and numbers $c_1, \ldots, c_n \in \mathbb{R}$,

$$\mathbb{E}(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n) = c_1 \mathbb{E}(X_1) + c_2 \mathbb{E}(X_2) + \cdots + c_n \mathbb{E}(X_n)$$

Linearity of expectation holds even for dependent random variables!

Indicator Random Variables

- An indicator random variable takes on two values: 0 and 1.
- Lemma. For an indicator random variable Y,

$$\mathbb{E}(Y) = \Pr(Y = 1).$$

Proof:

An indicator random variable Y for an event E is

$$Y = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Method for computing expectation of a random variable X:
Represent X as a sum of indicators and use linearity of expectation.

Top Hat question (Join Code: 413437)

You have a coin that shows HEADS with probability 3/4.
Let X be the number of HEADS in 1000 tosses of your coin.

You represent X as the sum: $X = X_1 + X_2 + \cdots + X_{1000}$.

What is X_1?

A. 3/4.

B. The number of HEADS.

C. The probability of HEADS in toss 1.

D. The number of heads in toss 1.

E. None of the above.

Top Hat question (Join Code: 413437)

You have a coin that shows HEADS with probability 3/4.
Let X be the number of HEADS in 1000 tosses of your coin.

What is the expectation of X?

A. 3/4.

B. 4/3.

C. 500.

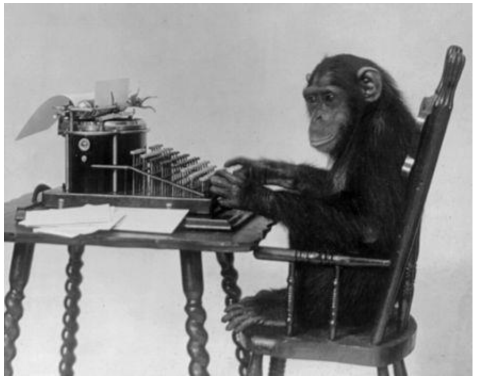
D. 750.

E. None of the above.

Example: Random Proofs

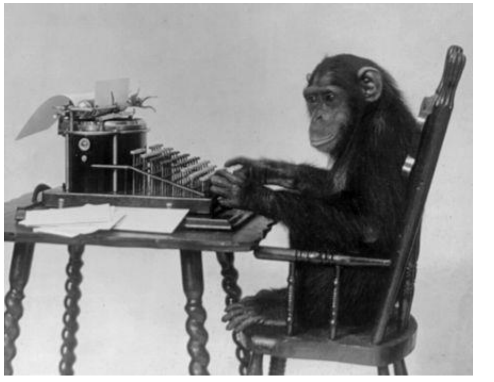
- Infinite Monkey Theorem. Given enough time, a hypothetical chimpanzee typing at random would, as part of its output, almost surely produce all of Shakespeare’s plays (or any other text.)

https://en.wikipedia.org/wiki/Infinite_monkey_theorem



- Suppose that the monkey types on a 26-letter keyboard
- Each letter is chosen independently and uniformly at random from the 26 letters
  - Suppose the monkey types $n = 1{,}000{,}000$ letters
  - Let X = number of times the sequence “proof” appears
  - What is $\mathbb{E}(X)$?

Example: Random Proofs



- Suppose the monkey types $n = 1{,}000{,}000$ letters
- Let X = number of times the sequence "proof" appears
- What is $\mathbb{E}(X)$?

Solution: For each position i in the sequence, let

Top Hat question (Join Code: 413437)

- Example: permutations
  - 3 students exchange their hats, so that everybody gets a random hat
  - Let X be the number of students that got their own hats.
  - E.g., if students 1,2,3 got hats 2,1,3 then X=1.
- PDF of X:

$$\Pr(X = 0) = \frac{1}{3}, \Pr(X = 1) = \frac{1}{2}, \Pr(X = 3) = \frac{1}{6}.$$

- What's the expectation of X?

A. 3/4

B. 1

C. 3/2

D. None of the above.

Example: Random Hats

- Example: permutations
  - n students exchange their hats, so that everybody gets a random hat
  - Let X be the number of students that got their own hats.
- What's the expectation of X for general n?

Functions of Random Variables

- A random variable can be a function of another random variable
- Examples: $Y = 3X + 1; Y = X^2; Y = |X|$
- If Y is a function of X then PDF of Y is related to the PDF of X
- Law of the Unconscious Statistician (LOTUS): If $Y = g(X)$ then

discrete: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_{x \in \mathbb{R}} g(x) \cdot f_X(x)$$

continuous: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

LOTUS: Examples

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- Example 1: Let X be uniform over $\{-4, \ldots, 4\}$. Then PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{9} & \text{if } x \in \{-4, \ldots, 4\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X^2) = \sum_{x \in \mathbb{R}} x^2 \cdot f_X(x) = \sum_{x \in \{-4,\ldots,4\}} x^2 \cdot \frac{1}{9} = \frac{2}{9}\left(1^2 + 2^2 + 3^2 + 4^2\right)$$

CS 237 LOTUS: Examples

- Law of the Unconscious Statistician (LOTUS): If $Y = g(X)$ then

discrete: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_{x \in \mathbb{R}} g(x) \cdot f_X(x)$$

continuous: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

- Example 2: Let X be uniform over $[0,1]$. Then PDF of X is

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$